Numerické modelování kompozitních vrstev a laminátů

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NUMERICAL MODELING OF COMPOSITE PLIES AND LAMINATES

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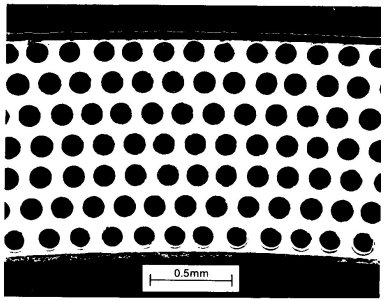
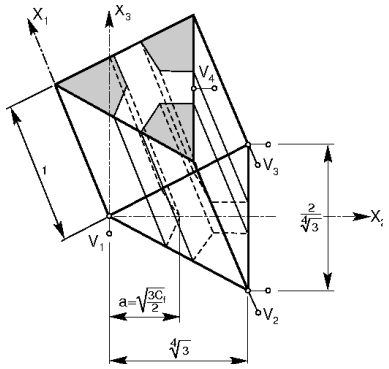
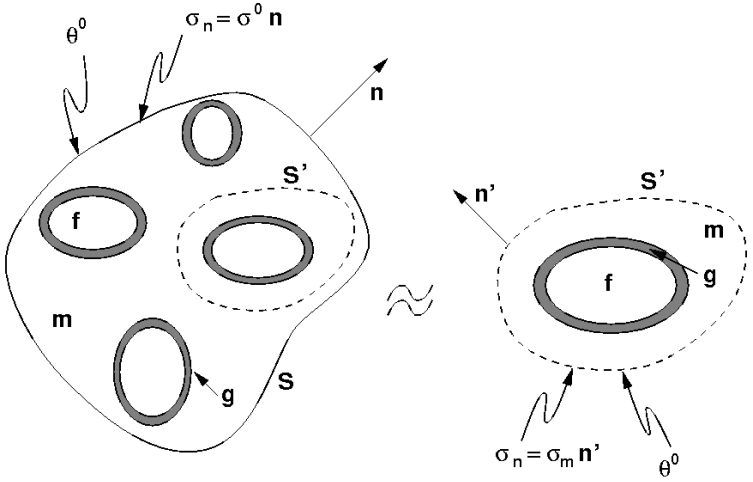
- Plies made of aligned fibers bonded to homogeneous matrix
- Derivation of local fields
  - Microstructural details are known $\longrightarrow$ FEM analysis assuming RVE in terms of PUC



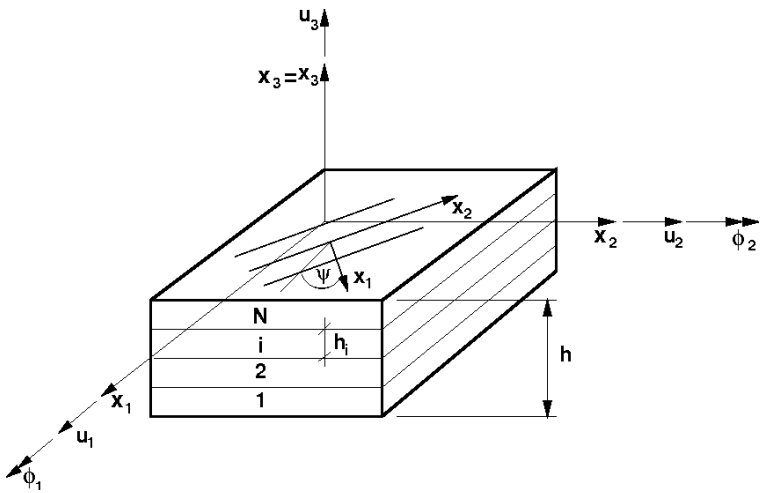


  - Volume fractions, shape and orientation of inclusions are given only $\longrightarrow$ Mori-Tanaka method

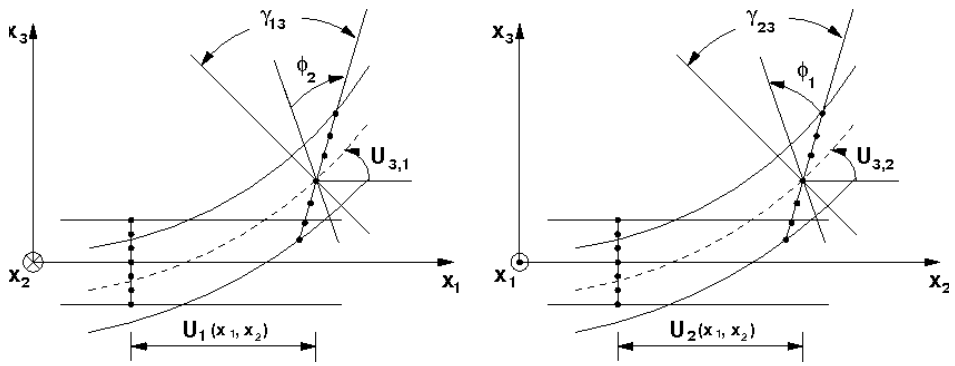


Classical laminated plate theory

Representative volume element



Kinematic assumptions



- Displacement field

$$
\begin{aligned}
u_1(x_1, x_2, x_3) &= U_1(x_1, x_2) + x_3\phi_2(x_1, x_2) \\
u_2(x_1, x_2, x_3) &= U_2(x_1, x_2) - x_3\phi_1(x_1, x_2) \\
u_3(x_1, x_2, x_3) &= U_3(x_1, x_2)
\end{aligned}
$$

- In-plane strains $\boldsymbol{E}_m = [E_{11},\ E_{22},\ E_{12}]^{\mathrm{T}}$

$$\begin{aligned} \boldsymbol{E}_m &= \boldsymbol{E}_m^0 + x_3 \mathbf{S}\boldsymbol{\kappa} \\ \boldsymbol{E}_m^0 &= \left[U_{1,1},\ U_{2,2},\ U_{1,2} + U_{2,1}\right]^{\mathrm{T}} \\ \boldsymbol{\kappa} &= \left[\phi_{1,2},\ \phi_{2,1},\ \phi_{1,1} - \phi_{2,2}\right]^{\mathrm{T}} \end{aligned}$$

- Out-of-plane strains $\boldsymbol{E}_s = [E_{23},\ E_{13}]^{\mathrm{T}}$

$$\boldsymbol{E}_s = \left[U_{3,2} - \phi_1,\ U_{3,1} + \phi_2\right]^{\mathrm{T}}$$

- Ply constitutive relations in local coordinates

$$\begin{aligned} \boldsymbol{\sigma}_m^i &= \mathbf{L}_m^i \left(\boldsymbol{\epsilon}_m^i - \boldsymbol{\mu}_m^i\right) & \boldsymbol{\sigma}_s^i &= \mathbf{L}_s^i \left(\boldsymbol{\epsilon}_s^i - \boldsymbol{\mu}_s^i\right) \\ \boldsymbol{\mu}_m^i &= \mathbf{m}_i \Delta\theta_i + \left(\boldsymbol{\epsilon}_m^{in}\right)_i & \boldsymbol{\mu}_s^i &= \left(\boldsymbol{\epsilon}_s^{in}\right)_i \end{aligned}$$

- Ply constitutive relations in global coordinates

$$\boldsymbol{S}_m^i = \overline{\mathbf{L}}_m^i \left(\boldsymbol{E}_m^i - \overline{\boldsymbol{\mu}}_m^i\right) \qquad \boldsymbol{S}_s^i = \overline{\mathbf{L}}_s^i \left(\boldsymbol{E}_s^i - \overline{\boldsymbol{\mu}}_s^i\right)$$

- Global fields in terms of their local counterparts

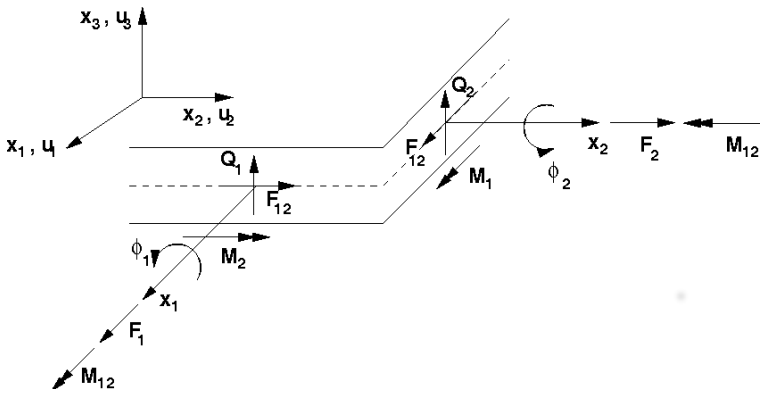
$$
\begin{aligned}
\boldsymbol{S}^i_m &= \mathbf{N}_i^{\mathrm{T}} \boldsymbol{\sigma}^i_m, & \boldsymbol{S}^i_s &= \tilde{\mathbf{N}}_i^{\mathrm{T}} \boldsymbol{\sigma}^i_s \\
\boldsymbol{E}^i_m &= \mathbf{R}_i^{\mathrm{T}} \boldsymbol{\epsilon}^i_m, & \boldsymbol{E}^i_s &= \tilde{\mathbf{R}}_i^{\mathrm{T}} \boldsymbol{\epsilon}^i_s \\
\overline{\boldsymbol{\mu}}^i_m &= \mathbf{R}_i^{\mathrm{T}} \boldsymbol{\mu}^i_m, & \overline{\boldsymbol{\mu}}^i_s &= \tilde{\mathbf{R}}_i^{\mathrm{T}} \boldsymbol{\mu}^i_s \\
\overline{\mathbf{L}}^i_m &= \mathbf{N}_i^{\mathrm{T}} \mathbf{L}^i_m \mathbf{N}_i & \overline{\mathbf{L}}^i_s &= \tilde{\mathbf{N}}_i^{\mathrm{T}} \mathbf{L}^i_s \tilde{\mathbf{N}}_i
\end{aligned}
$$

- Transformation matrices

$$
\mathbf{R}_i^{\mathrm{T}} = (\mathbf{N}_i)^{-1} = \begin{bmatrix} \cos^2\psi_i & \sin^2\psi_i & -\frac{1}{2}\sin 2\psi_i \\ \sin^2\psi_i & \cos^2\psi_i & \frac{1}{2}\sin 2\psi_i \\ \sin 2\psi_i & -\sin 2\psi_i & \cos 2\psi_i \end{bmatrix}
$$

$$
\tilde{\mathbf{R}}_i^{\mathrm{T}} = \tilde{\mathbf{N}}_i^{\mathrm{T}} = \begin{bmatrix} \cos\psi_i & \sin\psi_i \\ -\sin\psi_i & \cos\psi_i \end{bmatrix}
$$

- Force and moment resultants



$$\mathbf{F} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \boldsymbol{S}_m \mathrm{d}x_3 = \sum_{i=1}^{\mathrm{N}} \int_{-\frac{x_3^i}{2}}^{\frac{x_3^i}{2}} \boldsymbol{S}_m^i \mathrm{d}x_3$$

$$\mathbf{SM} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 \boldsymbol{S}_m \mathrm{d}x_3 = \sum_{i=1}^{\mathrm{N}} \int_{-\frac{x_3^i}{2}}^{\frac{x_3^i}{2}} x_3 \boldsymbol{S}_m^i \mathrm{d}x_3$$

$$\mathbf{Q} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \boldsymbol{S}_s \mathrm{d}x_3 = \sum_{i=1}^{\mathrm{N}} \int_{-\frac{x_3^i}{2}}^{\frac{x_3^i}{2}} \boldsymbol{S}_s^i \mathrm{d}x_3$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_{(3x3)} & x_3 \mathbf{I}_{(3x3)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{(2x2)} \end{bmatrix} \mathbf{Q}_i = \begin{bmatrix} \mathbf{I}_{(3x3)} & (x_3 - \bar{x}_3^i)\mathbf{I}_{(3x3)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{(2x2)} \end{bmatrix}$$

- Force and moment resultants in compact form $\mathcal{F} = [\mathbf{F}, \mathbf{SM}, \mathbf{Q}]^{\mathrm{T}}$

$$\begin{aligned}\mathcal{F} &= \sum_{i=1}^{\mathrm{N}} \overline{\mathbf{L}}_i \int_{-\frac{x_3^i}{2}}^{\frac{x_3^i}{2}} \mathbf{P}^{\mathrm{T}}\mathbf{P}\mathrm{d}x_3 \mathcal{E} + \sum_{i=1}^{\mathrm{N}} \int_{-\frac{x_3^i}{2}}^{\frac{x_3^i}{2}} \mathbf{P}^{\mathrm{T}}\mathbf{Q}_i \mathrm{d}x_3 \boldsymbol{\Lambda}_i \\ &= \mathcal{L}\mathcal{E} + \sum_{i=1}^{\mathrm{N}} \hat{\mathbf{Q}}_i \boldsymbol{\Lambda}_i = \mathcal{L}\mathcal{E} + \boldsymbol{\Lambda} \qquad (\boldsymbol{\lambda} = -\mathbf{L}\boldsymbol{\mu})\end{aligned}$$

$$\mathcal{L} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B} & \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{E} \end{bmatrix}, \quad \mathcal{E} = \left[\boldsymbol{E}_m^0, \mathbf{S}\boldsymbol{\kappa}, \boldsymbol{E}_s\right]^{\mathrm{T}}, \quad \boldsymbol{\Lambda}_i = \left[\overline{\boldsymbol{\lambda}}_m^i, \overline{\boldsymbol{\lambda}}_{m,3}^i, \overline{\boldsymbol{\lambda}}_s^i\right]^{\mathrm{T}}$$

$$\mathbf{A} = \sum_{i=1}^{\mathrm{N}} \left(x_3^{i-1} - x_3^i\right) \overline{\mathbf{L}}_m^i = \sum_{i=1}^{\mathrm{N}} h_i \overline{\mathbf{L}}_m^i, \quad \mathbf{B} = \sum_{i=1}^{\mathrm{N}} \frac{\left(x_3^{i-1}\right)^2 - \left(x_3^i\right)^2}{2} \overline{\mathbf{L}}_m^i$$

$$\mathbf{D} = \sum_{i=1}^{\mathrm{N}} \frac{\left(x_3^{i-1}\right)^3 - \left(x_3^i\right)^3}{3} \overline{\mathbf{L}}_m^i, \quad \mathbf{E} = \sum_{i=1}^{\mathrm{N}} \left(x_3^{i-1} - x_3^i\right) \overline{\mathbf{L}}_s^i = \sum_{i=1}^{\mathrm{N}} h_i \overline{\mathbf{L}}_s^i$$

- Ply stresses $\boldsymbol{S}_i = \left[\boldsymbol{S}_m^i, \boldsymbol{S}_s^i\right]^{\mathrm{T}}$ in terms of laminate fields

$$\boldsymbol{S}_i = \mathbf{H}_i \mathcal{F} + \sum_{j=1}^{\mathrm{N}} \mathbf{K}_{ij} \boldsymbol{\Lambda}_j$$

- Ply stress and eigenstress distribution factors

$$\begin{aligned}
\mathbf{H}_i &= \overline{\mathbf{L}}_i \mathbf{P} \mathcal{M} \\
\mathbf{K}_{ij} &= \delta_{ij} \mathbf{P} - \mathbf{H}_i \hat{\mathbf{Q}}_j \\
\boldsymbol{S}_i &= \overline{\mathbf{L}}_i \mathbf{P} \mathcal{M} \mathcal{F} - \overline{\mathbf{L}}_i \mathbf{P} \mathcal{M} \sum_{j=1}^{\mathrm{N}} \hat{\mathbf{Q}}_j \boldsymbol{\Lambda}_j + \mathbf{Q}_i \boldsymbol{\Lambda}_i
\end{aligned}$$

- Stresses in individual phases (fibers, matrix): Micromechanical models (MT method) combined with transformation field analysis (TFA)

Local stresses in individual phases

- Attention limited to two-phase fiber-matrix composites
- Individual phases are denoted by subscripts $\rho = f, m$ to identify fiber (f) and matrix (m) phases
- The load is assumed in terms of macroscopically uniform stresses $\boldsymbol{\sigma}_0 \leftarrow \boldsymbol{S_i}$, uniform temperature change $\Delta\theta$ and piece-wise uniform eigenstrains $\boldsymbol{\mu}_\rho$
- Phase stresses in the spirit of TFA

$$\boldsymbol{\sigma}_\rho(\boldsymbol{x}) = \mathbf{B}_\rho(\boldsymbol{x})\boldsymbol{\sigma}^0 + \mathbf{b}_\rho(\boldsymbol{x})\Delta\theta - \sum_{\eta=1}^{2} \mathbf{F}_{\rho\eta}(\boldsymbol{x})\mathbf{L}_\eta\boldsymbol{\mu}_\eta$$

  - $\mathbf{B}_\rho$ - stress concentration factor derived, e.g. from the Mori-Tanaka method [Benveniste (1987)]
  - $\mathbf{b}_\rho$ - thermal stress concentration factor [Laws (1973)]
  - $\mathbf{F}_{\rho\eta}$ - transformation influence function [Dvorak (1992)]

Two-phase composite

- thermal stress concentration factors

$$\mathbf{b}_\rho(\boldsymbol{x}) = [\mathbf{I} - \mathbf{B}_\rho(\boldsymbol{x})] \left(\mathbf{M}_m - \mathbf{M}_f\right)^{-1} \left(\mathbf{m}_f - \mathbf{m}_m\right)$$

- Transformation influence functions

$$\begin{aligned} \mathbf{F}_{\rho m}(\boldsymbol{x}) &= [\mathbf{I} - \mathbf{B}_\rho(\boldsymbol{x})] \left(\mathbf{M}_m - \mathbf{M}_f\right)^{-1} \mathbf{M}_m \\ \mathbf{F}_{\rho f}(\boldsymbol{x}) &= -[\mathbf{I} - \mathbf{B}_\rho(\boldsymbol{x})] \left(\mathbf{M}_m - \mathbf{M}_f\right)^{-1} \mathbf{M}_f \end{aligned}$$

- $\mathbf{m}_\rho$ - stores coefficients of thermal expansion of a given phase ρ
- $\mathbf{M}_\rho$ - phase compliance matrix
- NOTE: For a general number of phases the transformation influence functions must be computed numerically using a suitable RVE

- Basic assumptions:
  - Attention limited to symmetric laminates, e.g. $(0/\pm 45/90)_s$
  - Load applied through macroscopically uniform in-plane overall stresses $\boldsymbol{S} = [S_{11},\ S_{22},\ S_{12}]^T, (\boldsymbol{S} \leftarrow \boldsymbol{S}_m)$, uniform temperature change $\Delta\theta$ uniform in-plane ply eigenstrains $\boldsymbol{\mu}_i = [\mu_{11},\ \mu_{22},\ \mu_{12}]_i^{\mathrm{T}}$
- Overall response of composite laminate

$$\boldsymbol{S} = \overline{\mathbf{L}}\boldsymbol{E} + \overline{\mathbf{l}}\Delta\theta + \overline{\boldsymbol{\lambda}}, \quad \boldsymbol{E} = \overline{\mathbf{M}}\boldsymbol{S} + \overline{\mathbf{m}}\Delta\theta + \overline{\boldsymbol{\mu}}, \quad \overline{\mathbf{m}} = -\overline{\mathbf{L}}\overline{\mathbf{l}}, \quad \overline{\boldsymbol{\lambda}} = -\overline{\mathbf{L}}\overline{\boldsymbol{\mu}}$$

$$\overline{\mathbf{L}} = h\mathbf{A} = \sum_{i=1}^{\mathrm{N}} c_i\overline{\mathbf{L}}_i, \quad \overline{\mathbf{l}} = \sum_{i=1}^{\mathrm{N}} c_i\overline{\mathbf{l}}_i, \quad \overline{\boldsymbol{\lambda}} = \sum_{i=1}^{\mathrm{N}} c_i\overline{\boldsymbol{\lambda}}_i, \quad c_i = h_i/h$$

- Ply stresses from compatibility $\boldsymbol{E}_i = \boldsymbol{E}$ and equilibrium $\sum_{i=1}^{\mathrm{N}} \mathrm{c}_i\boldsymbol{S}_i = \boldsymbol{S}$ conditions, and Levin's formula

$$\boldsymbol{S}_i = \mathbf{H}_i\boldsymbol{S} + \mathbf{h}_i\Delta\theta - \sum_{j=1}^{\mathrm{N}} \mathbf{K}_{ij}\overline{\mathbf{L}}_j\overline{\boldsymbol{\mu}}_j$$

$$\mathbf{H}_i = \overline{\mathbf{L}}_i\overline{\mathbf{M}}, \quad \mathbf{K}_{ij} = \delta_{ij}\mathbf{I} - \mathrm{c}_j\mathbf{H}_i, \quad \mathbf{h}_i = \overline{\mathbf{l}}_i - \mathbf{H}_i\overline{\mathbf{l}} \quad \overline{\boldsymbol{\mu}}_j = \mathbf{R}_j^{\mathrm{T}} \sum_{\eta=1}^{2} c_\eta^j \mathbf{B}_\eta^{j\,\mathrm{T}} \boldsymbol{\mu}_\eta^j$$

Stresses in individual phases

- Recall phase stresses in the ith ply

$$\boldsymbol{\sigma}^i_\rho = \mathbf{B}^i_\rho \mathbf{R}_i \boldsymbol{S}_i + \mathbf{b}^i_\rho \Delta\theta - \sum_{\eta=1}^{\mathrm{M}} \mathbf{F}^i_{\rho\eta} \mathbf{L}^i_\eta \boldsymbol{\mu}^i_\eta$$

- Introducing the previous relations into the above equation gives

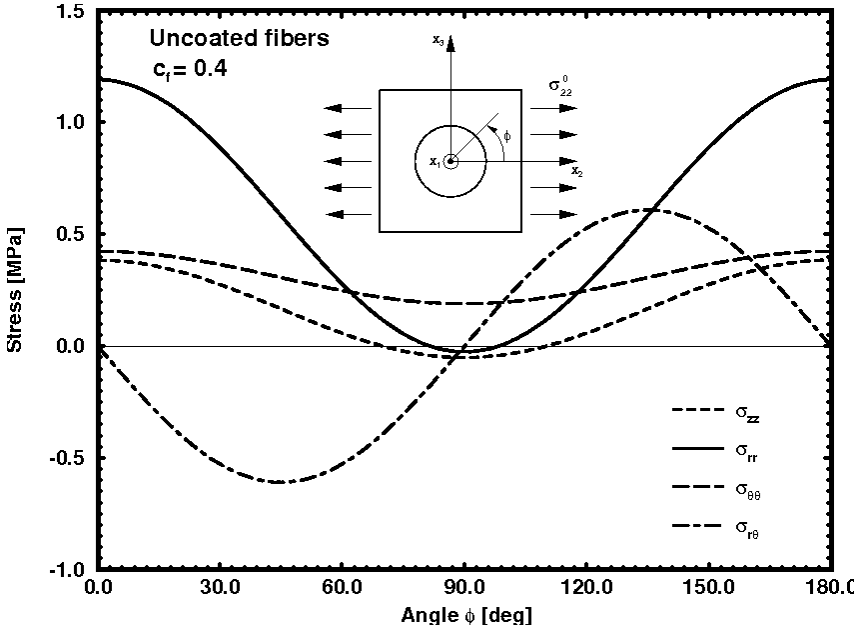
$$\begin{aligned} \boldsymbol{\sigma}^{\boldsymbol{i}}_{\boldsymbol{\rho}} &= \mathbf{B}^i_\rho \mathbf{R}_i \left(\mathbf{H}_i \boldsymbol{S} + \mathbf{h}_i \Delta\theta\right) + \mathbf{b}^i_\rho \Delta\theta - \\ & \sum_{\eta=1}^{\mathrm{M}} \mathbf{F}^i_{\rho\eta} \mathbf{L}^i_\eta \boldsymbol{\mu}^{\boldsymbol{i}}_{\boldsymbol{\eta}} - \mathbf{B}^i_\rho \mathbf{R}_i \sum_{j=1}^{\mathrm{N}} \mathbf{K}_{ij} \overline{\mathbf{L}}_j \mathbf{R}^{\mathrm{T}}_j \left[\sum_{\eta=1}^{\mathrm{M}} \bar{\mathrm{c}}^j_\eta \mathbf{B}^{j\ \mathrm{T}}_\eta \boldsymbol{\mu}^{\boldsymbol{j}}_{\boldsymbol{\eta}} \right] \end{aligned}$$

  - NOTE 1: the product $\mathbf{R}_i \boldsymbol{S}_i$ and the products $\mathbf{R}_i \left(\mathbf{H}_i \boldsymbol{S} + \mathbf{h}_i \Delta\theta\right)$ and $\mathbf{R}_i \sum_{j=1}^{\mathrm{N}} \mathbf{K}_{ij} \overline{\mathbf{L}}_j \overline{\mu}_j$ have to be augmented by zeros in rows 3,4 and 5 into (6x1) vectors to get correct phase averages $\boldsymbol{\sigma}^i_\rho$ as (6x1) vectors.
  - NOTE 2: The above equation can be now used to construct certain initial failure maps.

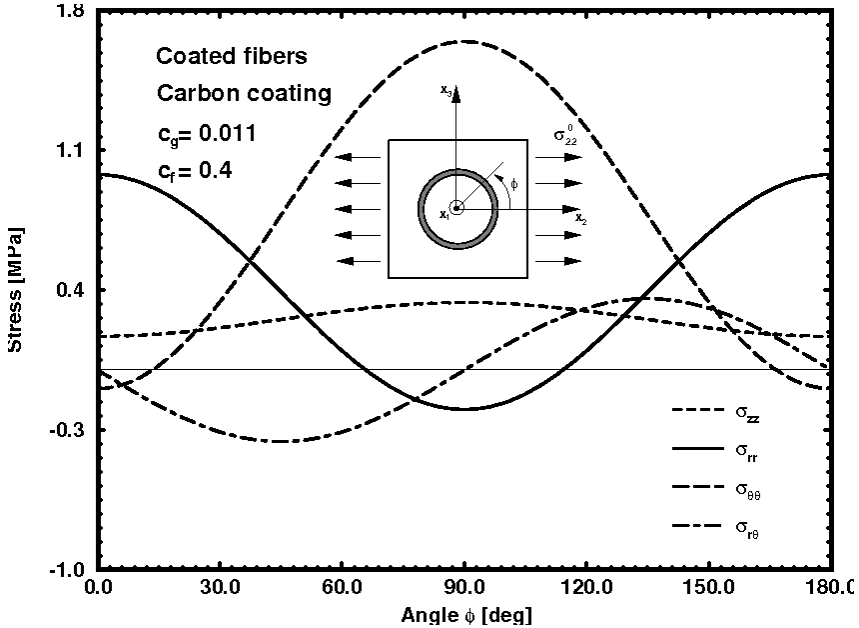
Local stresses in the fiber-matrix interface

- EXAMPLE 1: uniform transverse tension σ_{22}^0 applied to a unidirectional lamina of SCS-6/Ti_3Al

Uncoated fibers



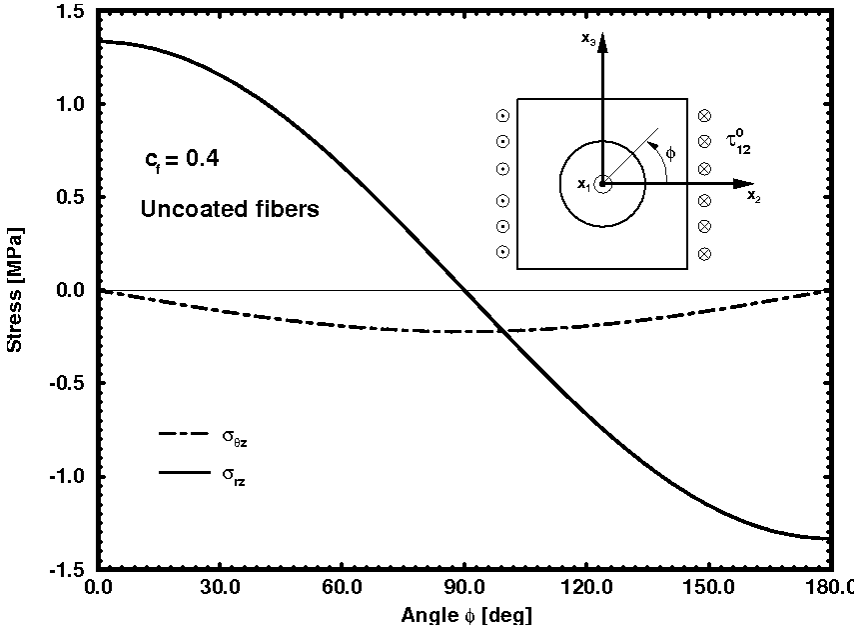
Coated fibers



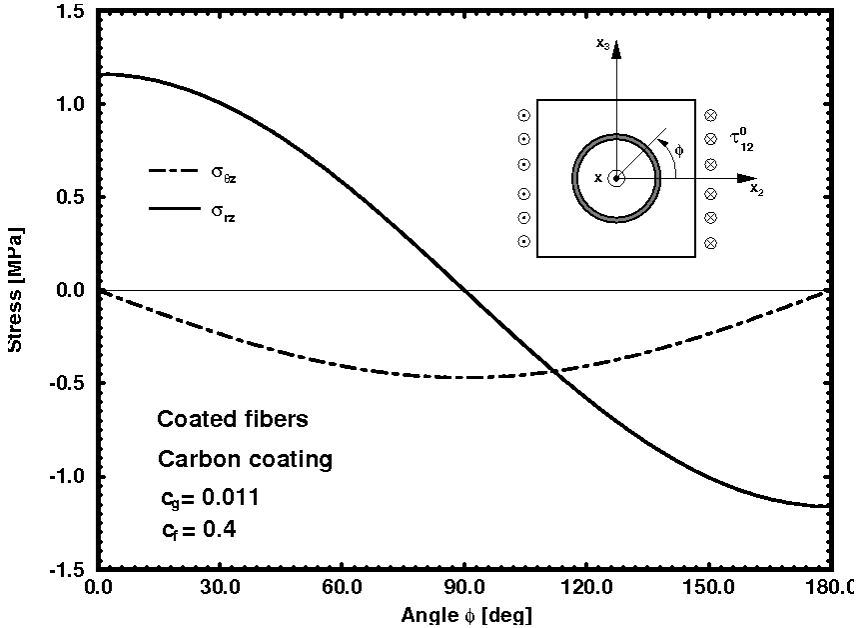
Local stresses in the fiber-matrix interface

- EXAMPLE 2: uniform out-of plane shear σ_{12}^0 applied to a unifrectional lamina of SCS-6/Ti_3Al

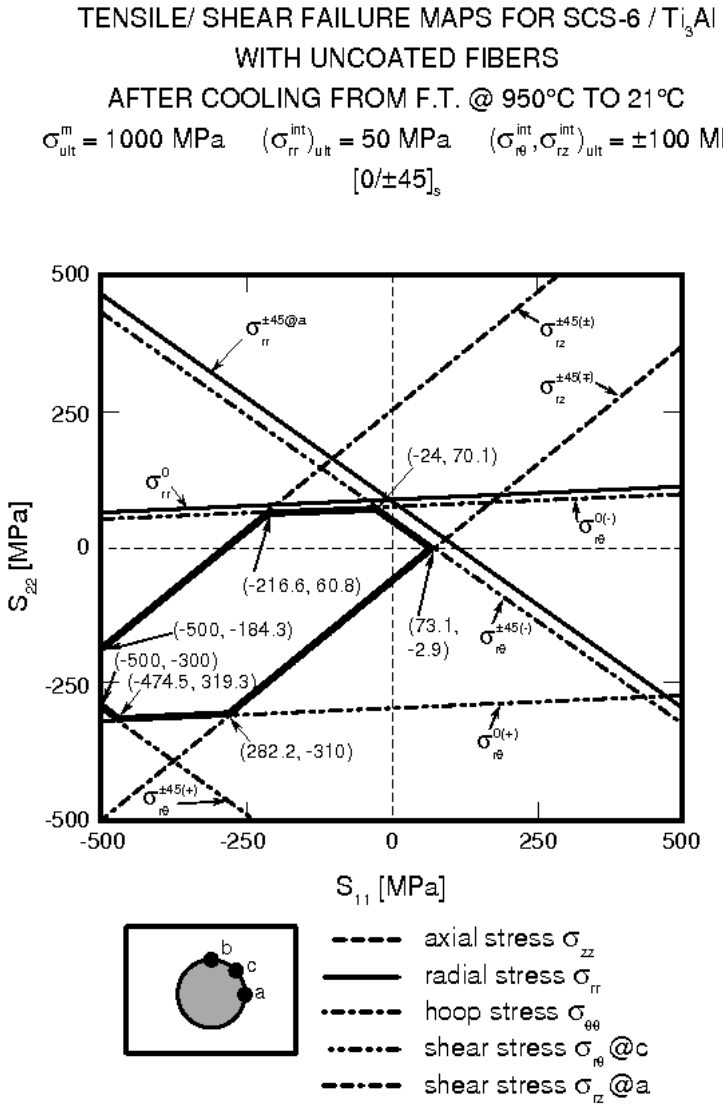
Uncoated fibers

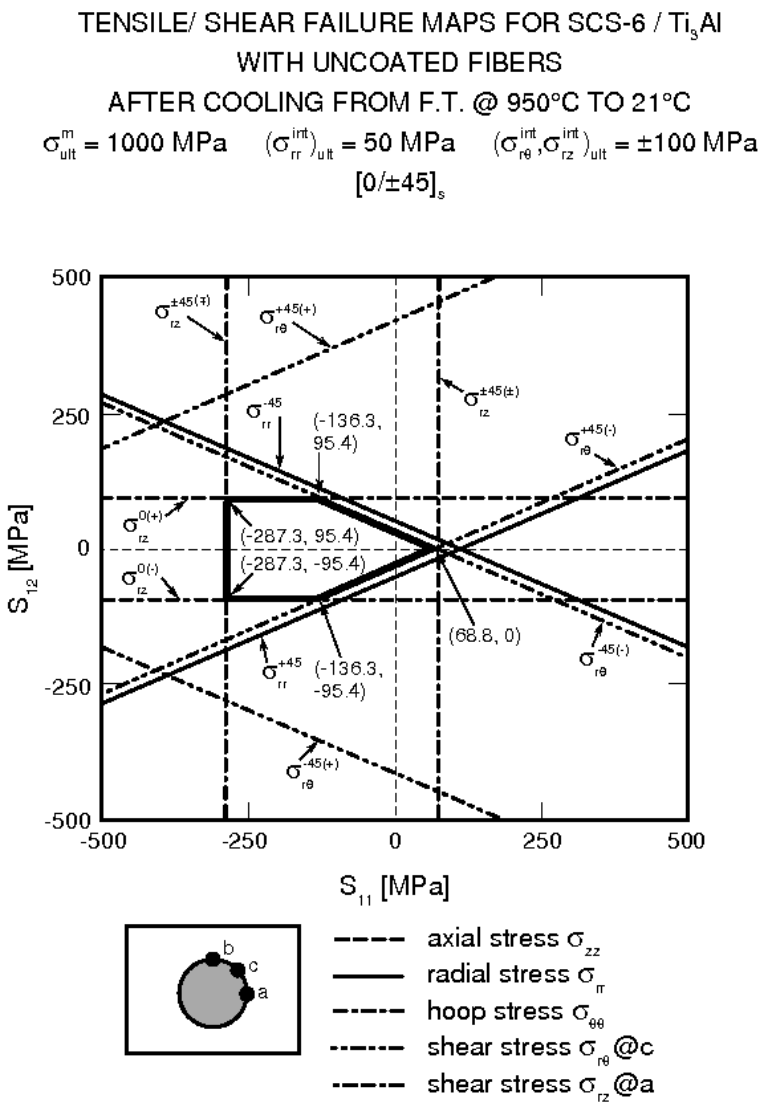


Coated fibers



SCS-6/Ti_3Al: $S_{11} - S_{22}$ and $S_{11} - S_{12}$ planes





SCS-6/Timetal-21: cooling followed by reheating

TENSILE FAILURE MAPS FOR SCS-6 / Timetal -21

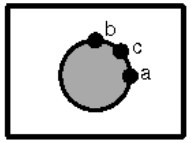
$[0/\pm45/90]_s$

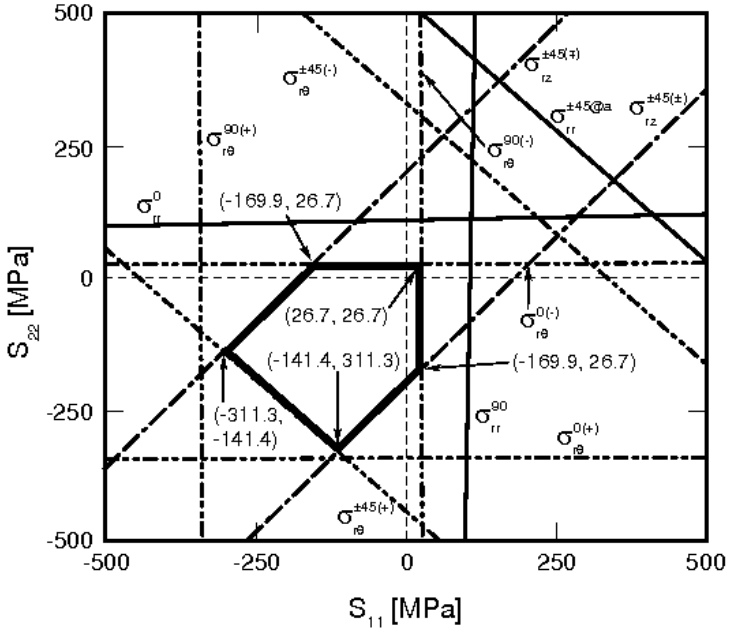
AFTER COOLING FROM F.T. @ 890°C TO 21°C

$\sigma^{m}_{ult} = \pm1000$ MPa $\quad (\sigma^{int}_{rr})_{ult} = 50$ MPa $\quad (\sigma^{int}_{r\theta}, \sigma^{int}_{rz})_{ult} = \pm100$ MPa

Local Residual Stresses @ 21°C

Stress [MPa]	0°/90°/-ply	±45°/-ply
σ^{m}_{zz}	475	477
$\sigma^{@a}_{rr}$	-61	-210
$\sigma^{@b}_{rr}$	-233	-239
$\sigma^{@c}_{r\theta}$	-86	-14.6





TENSILE/SHEAR FAILURE MAPS FOR SCS-6 / Timetal -21S

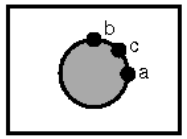
$[0/\pm45/90]_s$

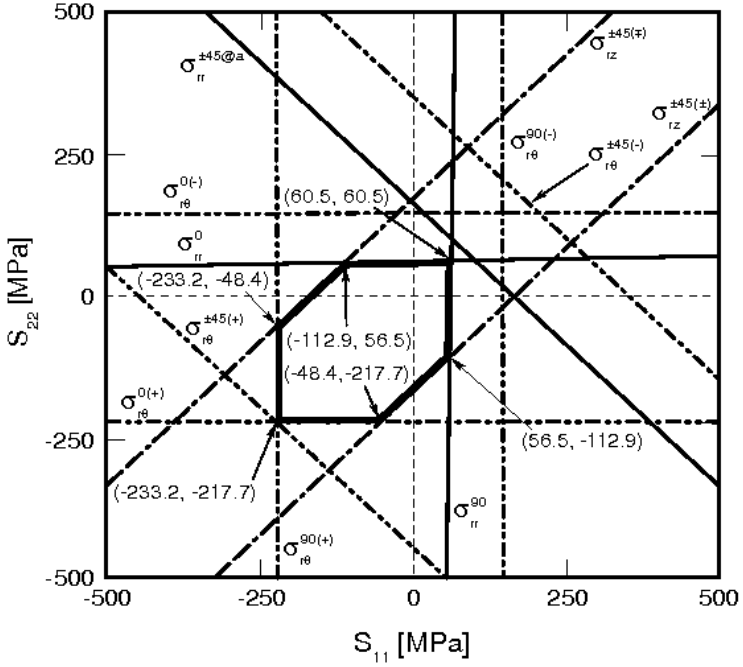
REHEATING FROM F.T. @ 21°C TO 650°C

$\sigma^{m}_{ult} = 1000$ MPa $\quad (\sigma^{int}_{rr})_{ult} = 50$ MPa $\quad (\sigma^{int}_{r\theta}, \sigma^{int}_{rz})_{ult} = \pm100$ MPa

Local Residual Stresses @ 650°C

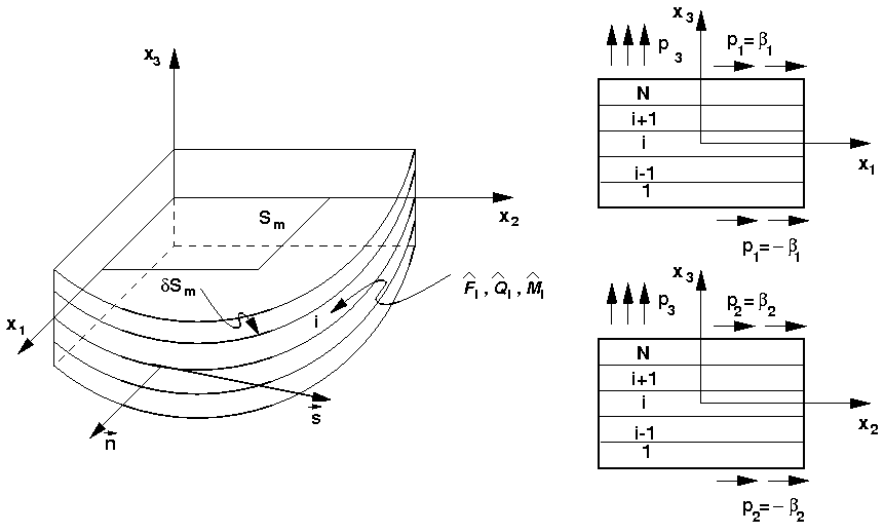
Stress [MPa]	0°/90°/-ply	±45°/-ply
σ^{m}_{zz}	96	98
$\sigma^{@a}_{rr}$	-12	-32
$\sigma^{@b}_{rr}$	-55	-57
$\sigma^{@c}_{r\theta}$	-21.5	-12.5



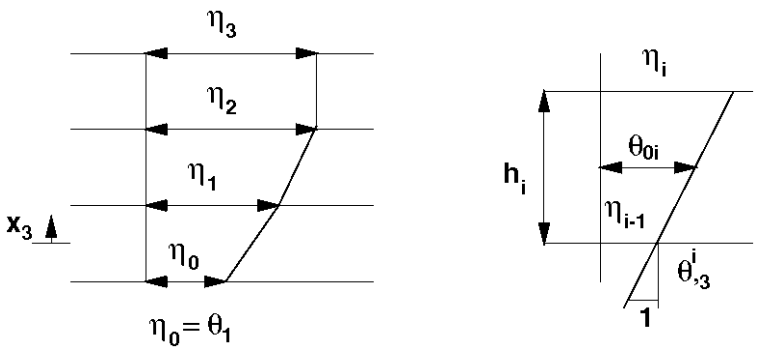


Layered theory for laminated plates Mau (1973)

Geometry and loading conditions



Temperature distribution



- Ply in-plane stresses in local coordinates

$$\boldsymbol{\sigma}_m^i = \mathbf{L}_m^i \left[\left(\epsilon_m^0 \right)_i - \mathbf{T}_F^i \Delta \boldsymbol{\eta}_i \right] + x_3^i \mathbf{L}_m^i \left(\mathbf{S} \boldsymbol{\kappa}_i - \mathbf{T}_D^i \Delta \boldsymbol{\eta}_i \right)$$

- Auxiliary matrices

$$\mathbf{T}_F^i = \begin{bmatrix} m_{11}^i & m_{11}^i \\ m_{22}^i & m_{22}^i \\ 0 & 0 \end{bmatrix}, \mathbf{T}_D^i = \frac{1}{h_i} \begin{bmatrix} m_{11}^i & -m_{22}^i \\ m_{11}^i & -m_{22}^i \\ 0 & 0 \end{bmatrix}, \quad \eta_i = [\Delta\eta_i, \Delta\eta_{i-1}]^{\mathrm{T}}$$

Formulation - kinematics

- Displacement field with each layer

$$
\begin{aligned}
u_1^i(x_1,x_2,x_3) &= U_1^i(x_1,x_2) + x_3^i\phi_2^i(x_1,x_2) \\
u_2^i(x_1,x_2,x_3) &= U_2^i(x_1,x_2) - x_3^i\phi_1^i(x_1,x_2) \\
u_3^i(x_1,x_2,x_3) &= U_3^i(x_1,x_2) \qquad\qquad i = 1,2,\ldots \mathrm{N}
\end{aligned}
$$

- Constraint (continuity) equations

$$
\begin{aligned}
g_1^i &= U_1^{i+1} - U_1^i - \frac{1}{2}\left[h^{i+1}\phi_2^{i+1} + h^i\phi_2^i\right] \\
g_2^i &= U_2^{i+1} - U_2^i + \frac{1}{2}\left[h^{i+1}\phi_1^{i+1} + h^i\phi_1^i\right] \qquad i = 1,2,\ldots \mathrm{N}-1
\end{aligned}
$$

- Principal of virtual work

$$
\delta W_{int} + \sum_{i=1}^{\mathrm{N}-1} \left[\beta_1\delta g_1 + \beta_2\delta g_2\right]_i = \delta W_{ext}
$$

Formulation - equilibrium

- Work done by internal forces

$$\delta W_{int} = \sum_{i=1}^{\mathrm{N}} \int_{S_m^i} \Big[\Big(F_1 \delta E_{m_1}^0 + F_2 \delta E_{m_2}^0 + F_{12} \delta E_{m_{12}}^0 \Big) + (M_1 \delta \kappa_1 + M_2 \delta \kappa_2 + M_{12} \delta \kappa_{12}) + (Q_1 \delta E_{s_{13}} + Q_2 \delta E_{s_{23}}) \Big]_i \, \mathrm{d}S_m$$

- Work done by applied load

$$\delta W_{ext} = \sum_{i=1}^{\mathrm{N}} \int_{\partial S_m^i} \sum_{i=1}^{\mathrm{N}} \Big(\hat{\mathrm{F}}_n \delta U_n + \hat{\mathrm{F}}_{ns} \delta U_s + \hat{\mathrm{Q}}_n \delta U_3 + \hat{\mathrm{M}}_n \delta \phi_s + \hat{\mathrm{M}}_{ns} \delta \phi_n \Big)_i \mathrm{d}$$

$$+ \int_{S_m^1} \left[p_1^- \delta u_1^1 \left(x_1, x_2, -\frac{h}{2} \right) + p_2^- \delta u_2^1 \left(x_1, x_2, -\frac{h}{2} \right) \right] \mathrm{d}S_m$$

$$+ \int_{S_m^{\mathrm{N}}} \left[p_1^+ \delta u_{\mathrm{N}}^1 \left(x_1, x_2, \frac{h}{2} \right) + p_2^+ \delta u_{\mathrm{N}}^1 \left(x_1, x_2, \frac{h}{2} \right) \right] \mathrm{d}S_m$$

$$+ \int_{S_m^i} p_3 \delta U_3 \mathrm{d}S_m$$

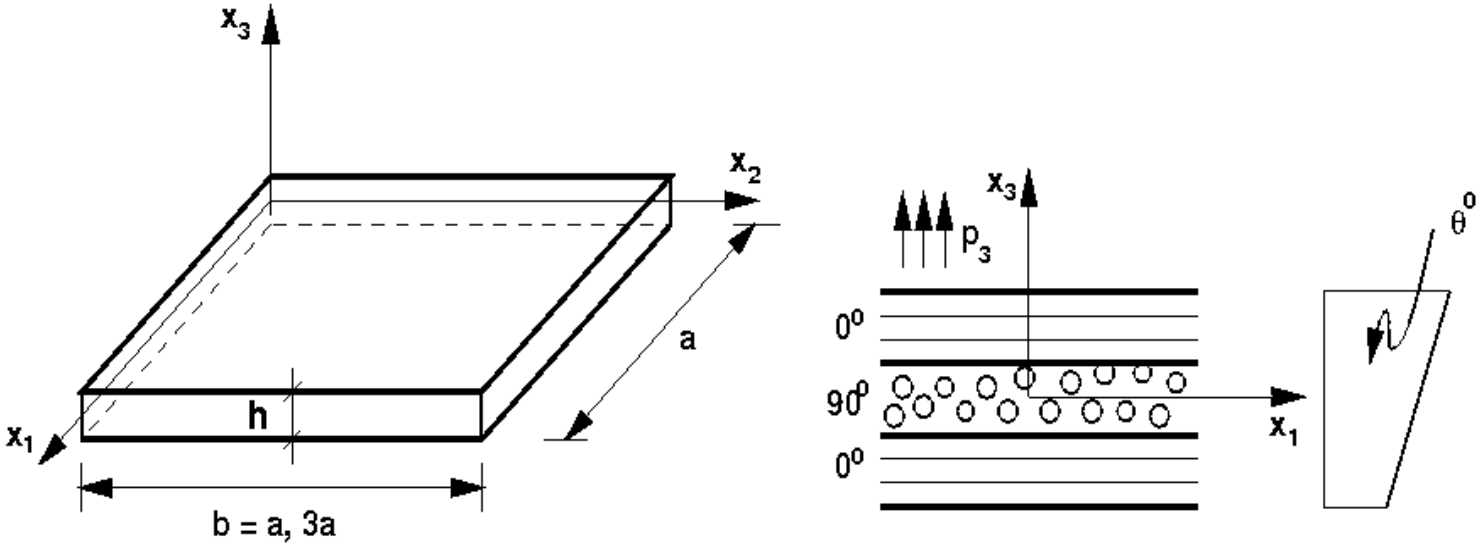
Formulation - equilibrium

- Work done by internal forces upon integration by parts

$$
\begin{aligned}
W_{int} = \sum_{i=1}^{\mathrm{N}} \Bigg\{ & \int_{S_m^i} \left[-\left(F_{1,1}^i + F_{12,2}^i\right)\delta U_1^i - \left(F_{12,1}^i + F_{2,2}^i\right)\delta U_2^i \right] \mathrm{d}S_m \\
& + \int_{S_m^i} \left[-\left(M_{1,2}^i + M_{12,1}^i\right)\delta\phi_1^i - \left(-M_{12,2}^i + M_{2,1}^i\right)\delta\phi_2^i \right] \mathrm{d}S_m \\
& + \int_{S_m^i} \left[-\left(Q_{1,1}^i + Q_{2,2}^i\right)\delta U_3 + Q_1^i\delta\phi_2^i - Q_2^i\delta\phi_1^i \right] \mathrm{d}S_m \\
& + \int_{\partial S_m^i} \left[F_n^i\delta U_n^i + F_{ns}^i\delta U_s^i + M_{ns}^i\delta\phi_n^i + M_n^i\delta\phi_s^i + Q_n^i\delta U_3 \right] \mathrm{d}s_m \Bigg\}
\end{aligned}
$$

- NOTE: Comparing terms in δE_{int} and δW_{ext} multiplied by the same virtual displacements provides equations of equilibrium for each layer and natural boundary conditions. For simplified geometries, these can be solved analytically using Fourier series.

Simply supported $(0^o/90^o/0^o)$ rectangular laminate



- Thermomechanical properties of Graphite/Epoxy system: stiffnesses in [GPa], coefficients of thermal expansion in $[10^{-6}/^0\mathrm{C}]$

E_L	E_T	G_L	G_T
172.4	6.9	3.5	1.4
ν_{LT}	ν_{TT}	α_L	α_T
0.25	0.25	0.0198	22.5

Simply supported $(0^o/90^o/0^o)$ rectangular laminate

- Temperature and loading via Fourier series

$$\Delta\eta^i = \sum_{mn} \Delta\eta^i_{mn} \sin\frac{m\pi x_1}{a} \sin\frac{n\pi x_2}{b}, \quad p_3 = \sum_{mn} p_{3_{mn}} \sin\frac{m\pi x_1}{a} \sin\frac{n\pi x_2}{b},$$

- Displacement field and constraints via Fourier series

$$U^i_1 = \sum_{mn} U^i_{1_{mn}} \cos\frac{m\pi x_1}{a} \sin\frac{n\pi x_2}{b}, \quad U^i_2 = \sum_{mn} U^i_{2_{mn}} \sin\frac{m\pi x_1}{a} \cos\frac{n\pi x_2}{b}$$

$$U_3 = \sum_{mn} U_{3_{mn}} \sin\frac{m\pi x_1}{a} \sin\frac{n\pi x_2}{b}$$

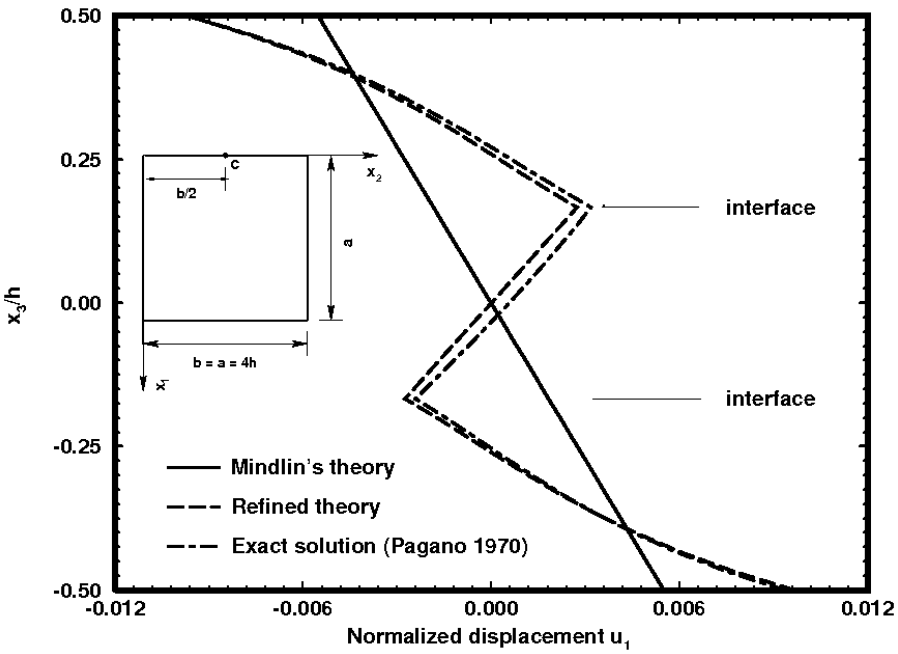
$$\phi^i_1 = \sum_{mn} \phi^i_{1_{mn}} \sin\frac{m\pi x_1}{a} \cos\frac{n\pi x_2}{b}, \quad \phi^i_2 = \sum_{mn} \phi^i_{2_{mn}} \cos\frac{m\pi x_1}{a} \sin\frac{n\pi x_2}{b}$$

$$\beta^i_1 = \sum_{mn} \beta^i_{1_{mn}} \cos\frac{m\pi x_1}{a} \sin\frac{n\pi x_2}{b}, \quad \beta^i_2 = \sum_{mn} \beta^i_{2_{mn}} \sin\frac{m\pi x_1}{a} \cos\frac{n\pi x_2}{b}$$

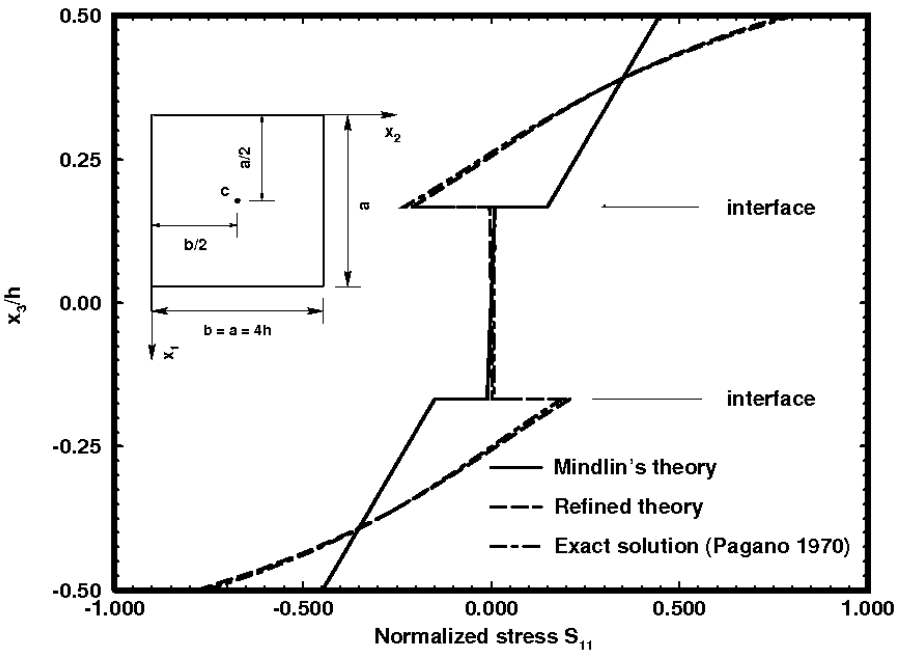
Simply supported $(0^o/90^o/0^o)$ rectangular laminate

- Comparison with Mindlin's bending theory and exact Pagano (1970) solution. Applied uniform pressure $p_3 = 1$ MPa.

Displacement $\overline{u}_1$



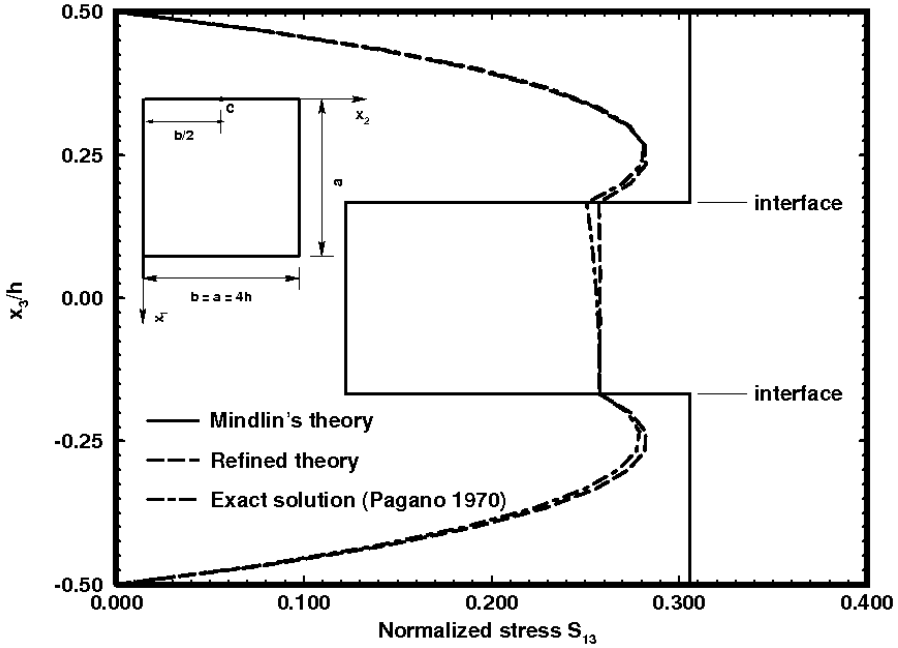
Stress $\overline{S}_{11}$



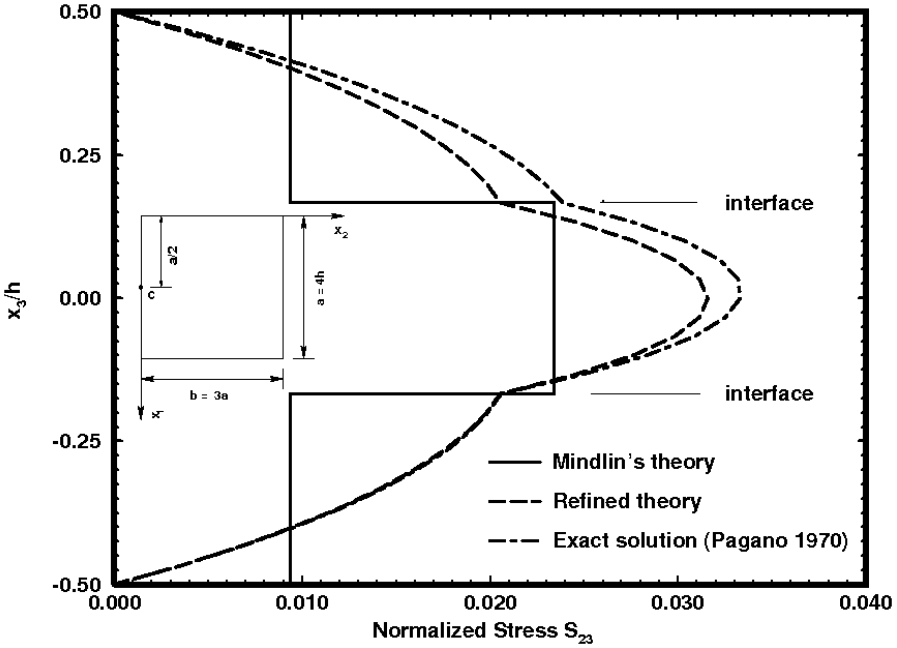
Simply supported $(0^o/90^o/0^o)$ rectangular laminate

- Comparison with Mindlin's bending theory and exact Pagano (1970) solution. Applied uniform pressure $p_3 = 1$ MPa.

Stress $\overline{S}_{13}$



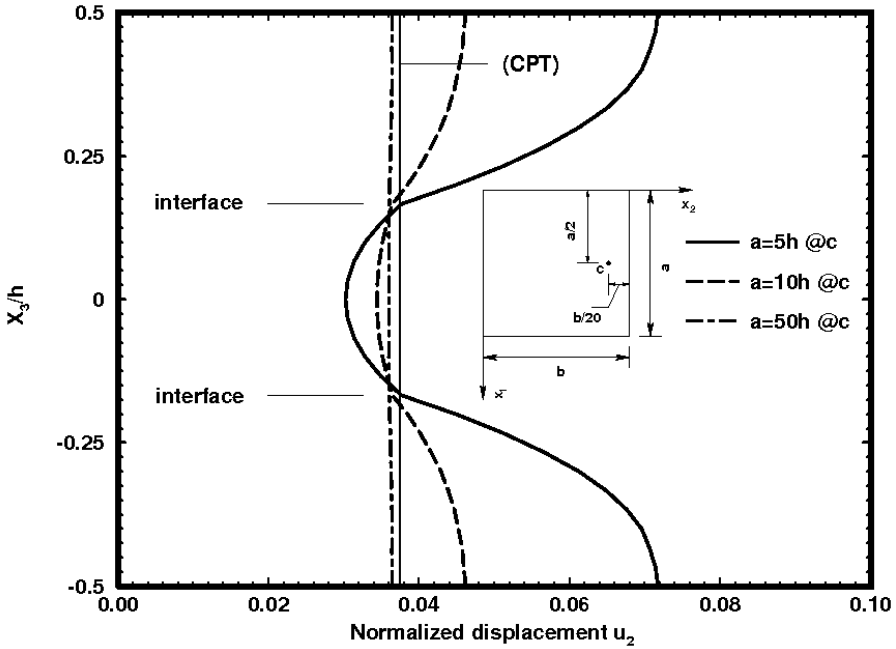
Stress $\overline{S}_{23}$



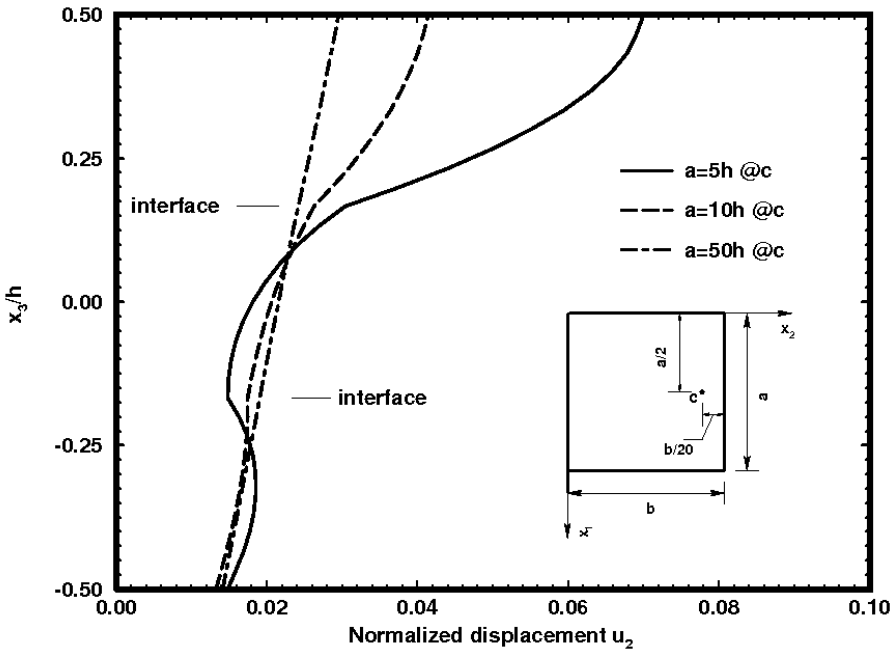
Simply supported $(0^o/90^o/0^o)$ rectangular laminate

- Applied temperature change: a) Uniform $\Delta = \theta = 100°$C, b) Linear $\Delta\theta^{top} = 100°$C, $\Delta\theta^{bottom} = 20°$C

Displacement $\overline{u}_2$ (a)



Displacement $\overline{u}_2$ (b)



Bending-stretching multilayered plate element

- Modified total potential energy functional Π_m

$$\Pi_m(\mathbf{u}, \boldsymbol{\beta}) = U_{int} + \int_{S_m} \sum_{i=1}^{\mathrm{N}-1} (\boldsymbol{\beta_i})^{\mathrm{T}} \boldsymbol{g}_i \, \mathrm{d}S - U_{ext}$$

- Internal Energy

$$\begin{aligned} U_{int} &= \frac{1}{2} \int_{S_m} \sum_{i=1}^{\mathrm{N}} \Big[\left(\boldsymbol{E}_m^0 - \overline{\mathbf{T}}_F \Delta\boldsymbol{\eta} \right)^{\mathrm{T}} \mathbf{A} \left(\boldsymbol{E}_m^0 - \overline{\mathbf{T}}_F \Delta\boldsymbol{\eta} \right) \\ &\quad + \left(\mathbf{S}\boldsymbol{\kappa} - \overline{\mathbf{F}}_D \Delta\boldsymbol{\eta} \right)^{\mathrm{T}} \mathbf{D} \left(\mathbf{S}\boldsymbol{\kappa} - \overline{\mathbf{T}}_D \Delta\boldsymbol{\eta} \right) + (\boldsymbol{E}_s)^{\mathrm{T}} \mathbf{C} \boldsymbol{E}_s \Big]_i \mathrm{d}S \end{aligned}$$

- Displacement field

$$\langle u_1, u_2 \rangle_i^{\mathrm{T}} = \boldsymbol{N}_1 \boldsymbol{d}_u^i + x_3^i \boldsymbol{N}_1 \boldsymbol{d}_\phi^i, \quad u_3 = \boldsymbol{N}_2 \boldsymbol{d}_w$$

- Strain field

$$\left(\boldsymbol{E}_m^0 \right)_i = \boldsymbol{B}_u \boldsymbol{d}_u^i, \quad \boldsymbol{\kappa}_i = \boldsymbol{B}_b \boldsymbol{d}_\phi^i, \quad \boldsymbol{E}_s^i = \boldsymbol{B}_{s1} \boldsymbol{d}_\phi^i + \boldsymbol{B}_{s2} \boldsymbol{d}_w$$

Bending-stretching multilayered plate element

- Discrete form of constraint conditions

$$\boldsymbol{g}_i = \boldsymbol{G}_i \boldsymbol{M} \boldsymbol{d}_g^i, \quad \left(\boldsymbol{d}_g^i\right)^{\mathrm{T}} = \left\langle \left(\boldsymbol{d}_u^i\right)^{\mathrm{T}}, \left(\boldsymbol{d}_\phi^i\right)^{\mathrm{T}}, \left(\boldsymbol{d}_u^{i+1}\right)^{\mathrm{T}}, \left(\boldsymbol{d}_\phi^{i+1}\right)^{\mathrm{T}} \right\rangle$$

- Matrix $\boldsymbol{M}$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{N} & 0 & 0 & 0 \\ 0 & \boldsymbol{N} & 0 & 0 \\ 0 & 0 & \boldsymbol{N} & 0 \\ 0 & 0 & 0 & \boldsymbol{N} \end{bmatrix}, \quad \boldsymbol{N} = [N_1, N_2, \ldots N_{n_{en}}]$$

- Matrix $\boldsymbol{G}_i = [\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{G}_3, \boldsymbol{G}_4]_i$

$$\boldsymbol{G}_1^i = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \boldsymbol{G}_2^i = \begin{bmatrix} 0 & -\frac{h_i}{2} \\ \frac{h_i}{2} & 0 \end{bmatrix}$$

$$\boldsymbol{G}_3^i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{G}_4^i = \begin{bmatrix} 0 & -\frac{h_{i+1}}{2} \\ \frac{h_{i+1}}{2} & 0 \end{bmatrix}$$

Bending-stretching multilayered plate element

$$
\begin{aligned}
\Pi_m(\boldsymbol{d},\beta) \;=\; & \sum_{n_{el}} \Bigg\{ \sum_{i=1}^{\mathrm{N}} \boldsymbol{d}_i^{\mathrm{T}} \boldsymbol{K}_i \boldsymbol{d}_i \,\mathrm{d}S + \int_{S_m} \sum_{i=1}^{\mathrm{N}-1} (\beta)_i^T \, \boldsymbol{G}_i M d_g^i \,\mathrm{d}S \\
& - \int_{S_m} \sum_{i=1}^{\mathrm{N}} \left[\boldsymbol{d}_u^{\mathrm{T}} \boldsymbol{B}_u^{\mathrm{T}} \mathbf{A} \overline{\mathbf{T}}_F \Delta\boldsymbol{\eta} + \boldsymbol{d}_\phi^{\mathrm{T}} \boldsymbol{B}_\phi^{\mathrm{T}} \mathbf{S}^{\mathrm{T}} \mathbf{D}_i \overline{\mathbf{T}}_D \Delta\boldsymbol{\eta} \right]_i \mathrm{d}S \\
& - \int_{S_m} \left[\boldsymbol{d}_u^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{p}^- - \frac{h}{2} \boldsymbol{d}_\phi^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{p}^- \right]_1 \mathrm{d}S \\
& - \int_{S_m} \left[\boldsymbol{d}_u^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{p}^+ - \frac{h}{2} \boldsymbol{d}_\phi^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{p}^+ \right]_{\mathrm{N}} \mathrm{d}S - \int_{S_m} \boldsymbol{d}_w^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} p_3 \,\mathrm{d}S \\
& - \int_{\partial S_m^i} \sum_{i=1}^{\mathrm{N}} \Big[\boldsymbol{d}_{un}^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} \hat{\mathrm{F}}_n + \boldsymbol{d}_{us}^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} \hat{\mathrm{F}}_{ns} \\
& + \boldsymbol{d}_w^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} \hat{\mathrm{Q}}_n + \boldsymbol{d}_{\phi n}^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} \hat{\mathrm{M}}_n + \boldsymbol{d}_{\phi s}^{\mathrm{T}} \boldsymbol{N}^{\mathrm{T}} \hat{\mathrm{M}}_{ns} \Big]_i \Bigg\}
\end{aligned}
$$

Bending-stretching multilayered plate element

- The stationary conditions with respect to nodal parameters $\boldsymbol{d}$ and Lagrange multipliers $\boldsymbol{\beta}$, i.e. $\partial\Pi_m/(\partial\boldsymbol{d}\,\partial\boldsymbol{\beta}) = 0$, yields the system of equilibrium and constraint conditions

$$\begin{aligned} \boldsymbol{K}\boldsymbol{d} + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\beta} &= \boldsymbol{R} \\ \boldsymbol{Q}\boldsymbol{d} &= \boldsymbol{0} \end{aligned}$$

- Single layered element k

$$\boldsymbol{d}_k^{\mathrm{T}} = \left\langle \left(\boldsymbol{d}_u^1\right)^{\mathrm{T}}, \left(\boldsymbol{d}_\phi^1\right)^{\mathrm{T}}, \ldots \left(\boldsymbol{d}_u^i\right)^{\mathrm{T}}, \left(\boldsymbol{d}_\phi^i\right)^{\mathrm{T}}, \ldots \left(\boldsymbol{d}_u^{\mathrm{N}}\right)^{\mathrm{T}}, \left(\boldsymbol{d}_\phi^{\mathrm{N}}\right)^{\mathrm{T}}, \boldsymbol{d}_w^{\mathrm{T}} \right\rangle$$

$$\begin{bmatrix} \boldsymbol{k}_k & \boldsymbol{Q}_k^{\mathrm{T}} \\ \boldsymbol{Q}_k & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_k \\ \boldsymbol{\beta}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_k \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{k}_k^i = \begin{bmatrix} \boldsymbol{k}_u & 0 & 0 \\ 0 & \boldsymbol{k}_b + \boldsymbol{k}_{s1} & \boldsymbol{k}_{s2} \\ 0 & \boldsymbol{k}_{s2}^{\mathrm{T}} & \boldsymbol{k}_{s3} \end{bmatrix}_i$$

- NOTE: Reduced integration is used for submatrices associated with out-of-plane shear to avoid shear locking

Bending-stretching multilayered plate element

- Example of three-layered plate element

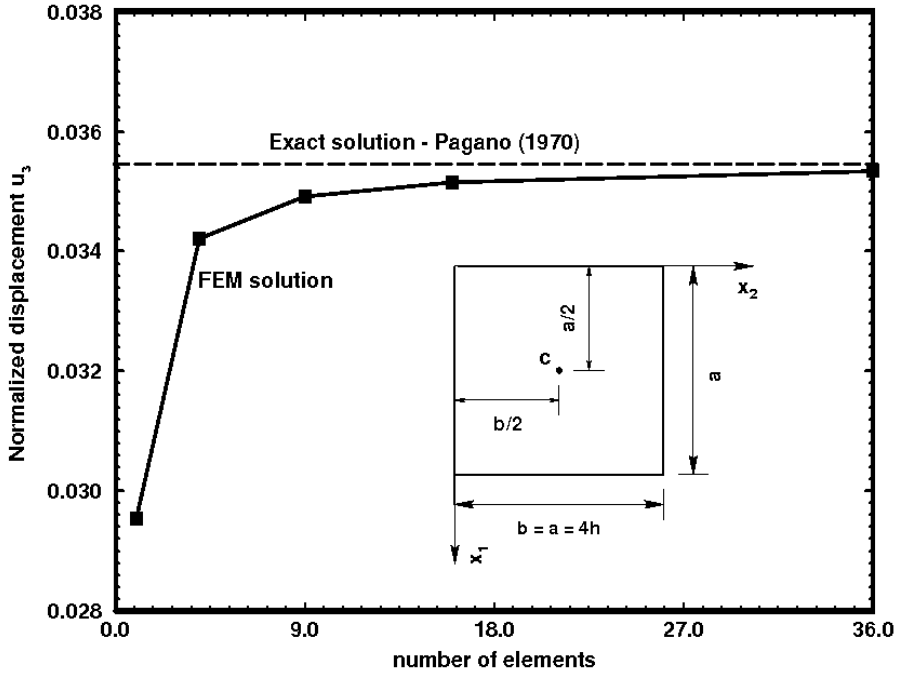
$$
\begin{bmatrix}
\boldsymbol{k}_u^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{g}}_1^1 & \mathbf{0} \\
\mathbf{0} & \boldsymbol{k}_b^1 + \boldsymbol{k}_{s1}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{k}_{s2}^1 & \overline{\boldsymbol{g}}_2^1 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \boldsymbol{k}_u^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{g}}_3^1 & \overline{\boldsymbol{g}}_1^2 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{k}_b^2 + \boldsymbol{k}_{s1}^2 & \mathbf{0} & \mathbf{0} & \boldsymbol{k}_{s2}^2 & \overline{\boldsymbol{g}}_4^1 & \overline{\boldsymbol{g}}_2^2 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{k}_u^3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{g}}_3^2 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{k}_b^3 + \boldsymbol{k}_{s1}^3 & \boldsymbol{k}_{s2}^3 & \mathbf{0} & \overline{\boldsymbol{g}}_4^2 \\
\mathbf{0} & \left(\boldsymbol{k}_{s2}^1\right)^{\mathrm{T}} & \mathbf{0} & \left(\boldsymbol{k}_{s2}^2\right)^{\mathrm{T}} & \mathbf{0} & \left(\boldsymbol{k}_{s2}^3\right)^{\mathrm{T}} & \sum_{i=1}^{\mathrm{N}} \boldsymbol{k}_{s3}^i & \mathbf{0} & \mathbf{0} \\
\left(\overline{\boldsymbol{g}}_1^1\right)^{\mathrm{T}} & \left(\overline{\boldsymbol{g}}_2^1\right)^{\mathrm{T}} & \left(\overline{\boldsymbol{g}}_3^1\right)^{\mathrm{T}} & \left(\overline{\boldsymbol{g}}_4^1\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \left(\overline{\boldsymbol{g}}_1^2\right)^{\mathrm{T}} & \left(\overline{\boldsymbol{g}}_2^2\right)^{\mathrm{T}} & \left(\overline{\boldsymbol{g}}_3^2\right)^{\mathrm{T}} & \left(\overline{\boldsymbol{g}}_4^2\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix}
$$

- Matrix $\overline{\boldsymbol{g}}_j^i$

$$
\overline{\boldsymbol{g}}_j^i = \int_{S_m} \boldsymbol{N}^{\mathrm{T}} \left(\boldsymbol{G}_j^i\right)^{\mathrm{T}} \mathrm{d}S, \quad j = 1, 2, \ldots 4
$$

EXAMPLE 1: $(0^o/90^o/0^o)$ rectangular laminate

- Convergence of $\overline{u}_3$ due to $p_3 = 1$ MPa.



- Normalized quantities

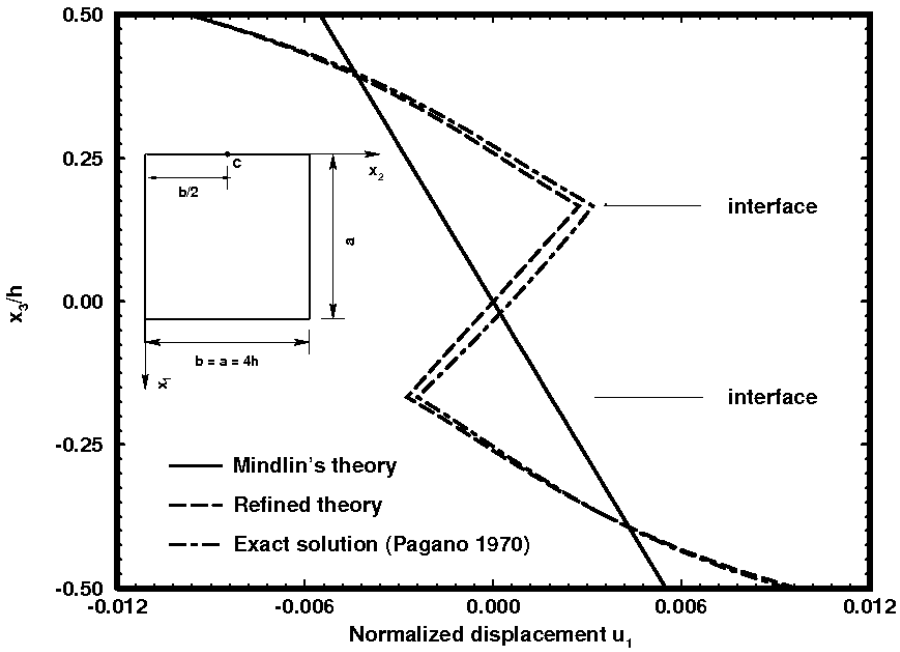
$$(\bar{S}_{11}, \bar{S}_{22}, \bar{S}_{12}) = \frac{1}{p_3 S^2}(S_{11}, S_{22}, S_{12})$$

$$(\bar{S}_{13}, \bar{S}_{23}) = \frac{1}{p_3 S}(S_{13}, S_{23}) \quad \bar{u} = \frac{E_T u}{p_3 h S^3}, \qquad S = \frac{a}{h}$$

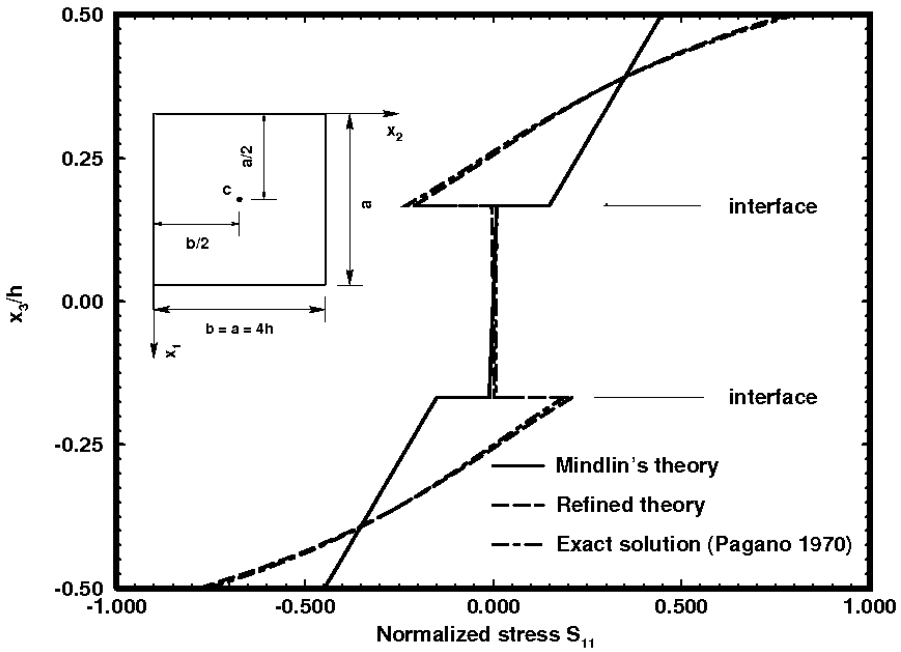
EXAMPLE 1: $(0^o/90^o/0^o)$ rectangular laminate

- Comparison with exact Pagano (1970) solution. Applied uniform pressure $p_3 = 1$ MPa. Each lamina subdivided into 3 additional plies.

Displacement $\overline{u}_1$



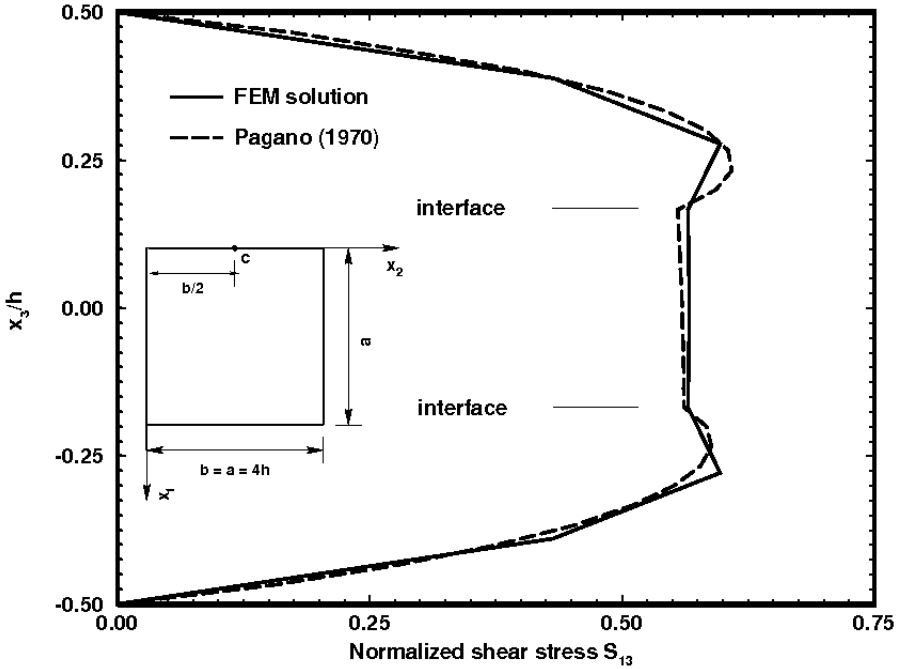
Stress $\overline{S}_{11}$



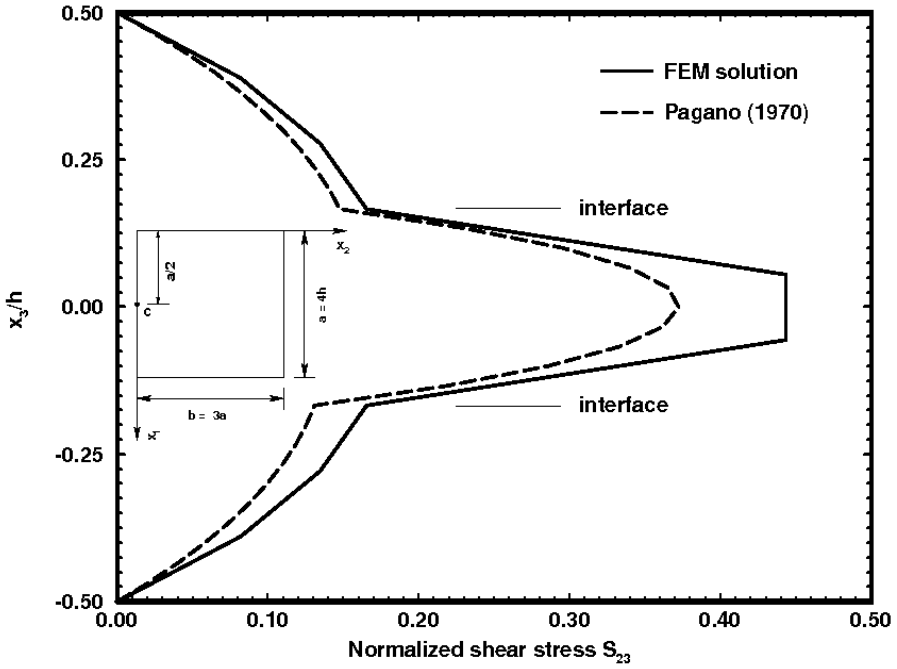
EXAMPLE 1: $(0^o/90^o/0^o)$ rectangular laminate

- Comparison with exact Pagano (1970) solution. Applied uniform pressure $p_3 = 1$ MPa. Each lamina subdivided into 3 additional plies.

Stress $\overline{S}_{13}$

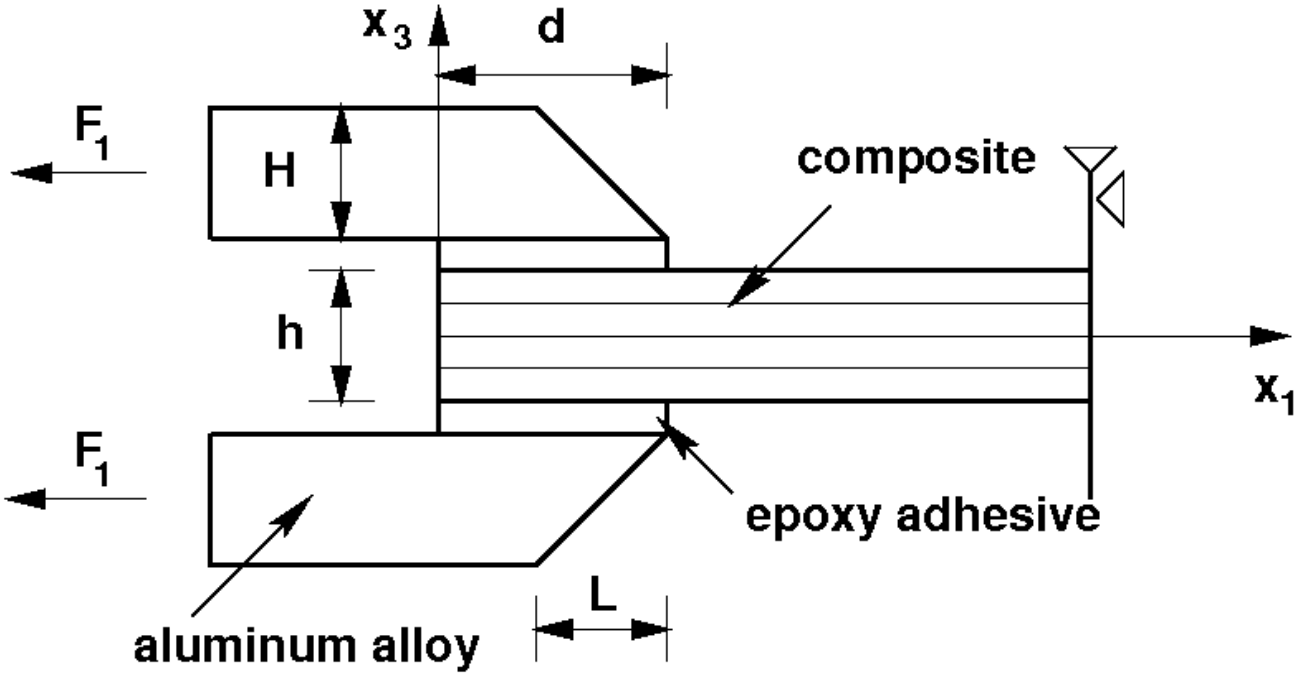


Stress $\overline{S}_{23}$



EXAMPLE 2: Application to the analysis of lap-joints

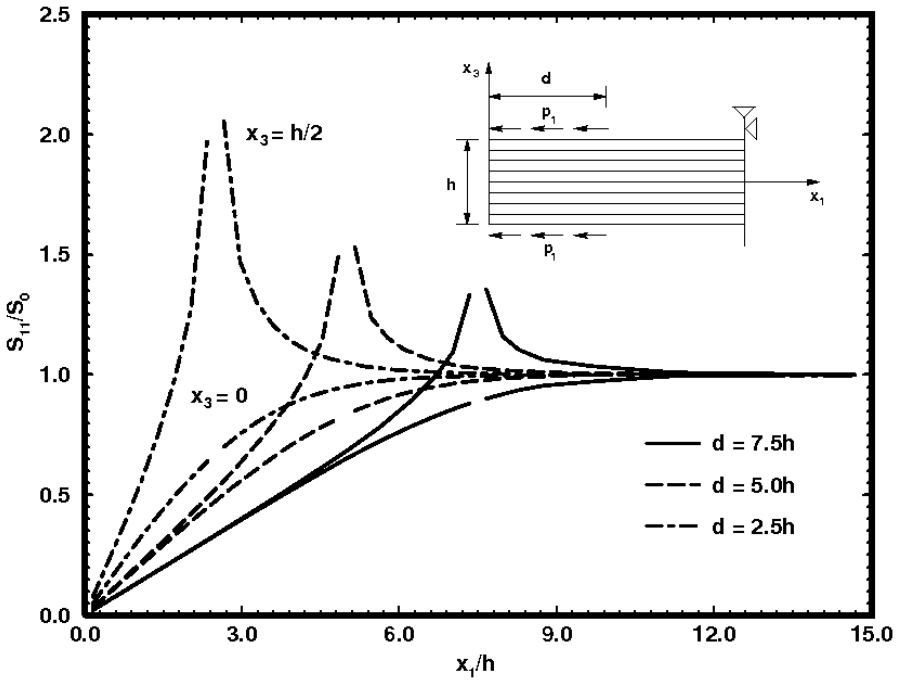
- Tensile failure of adhesively bonded composite laminates
- Graphite-Epoxy $[0]_N$ laminate subjected to uniform shear
- Interface element used to represent zero-thickness adhesive
- The effect of overlap length, d/h and L/d ratio examined
- Geometry and loading conditions of typical lap-joint



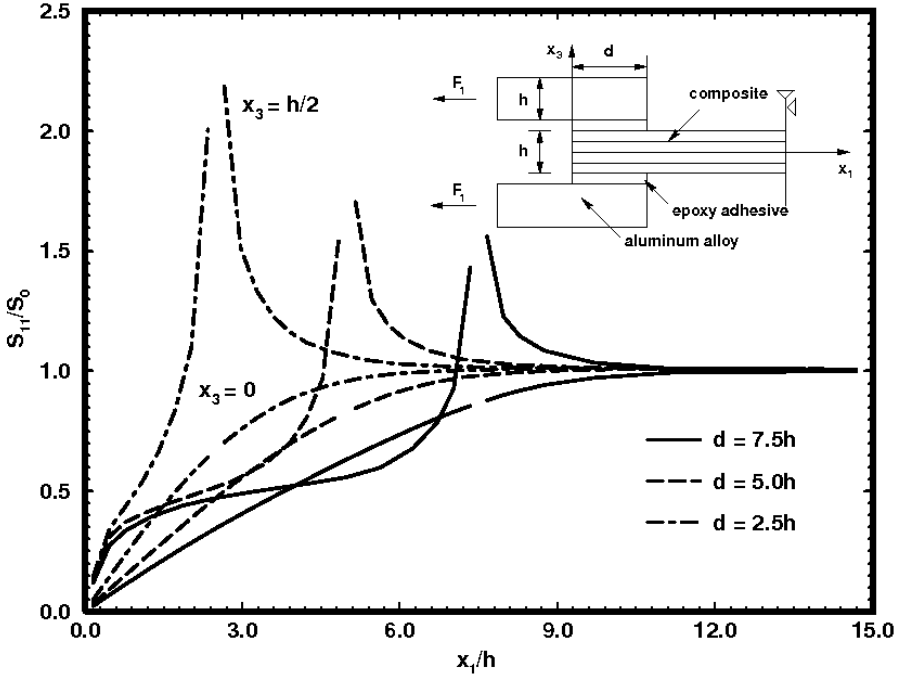
EXAMPLE 2: Application to the analysis of lap-joints

- Influence of d/h ratio

Tensile tress $\overline{S}_{11}, p = 2F/d$



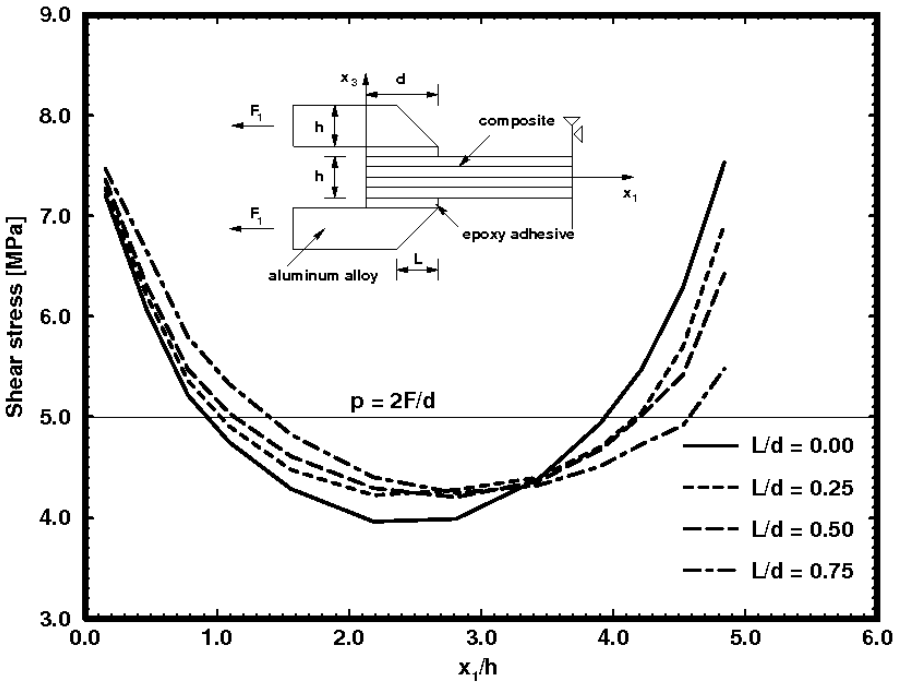
Tensile stress $\overline{S}_{11}$



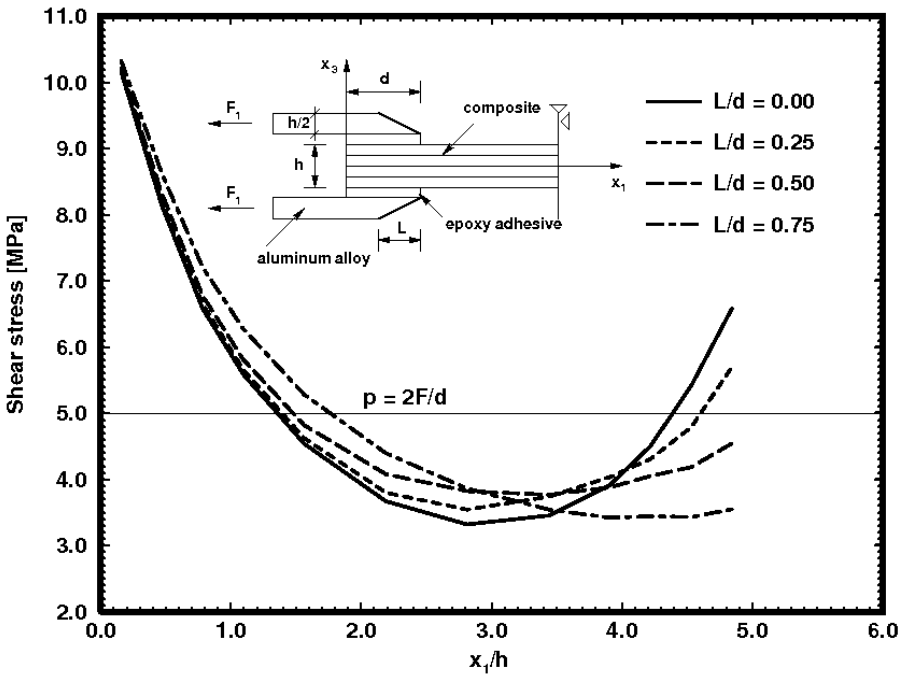
EXAMPLE 2: Application to the analysis of lap-joints

- Influence of adherent thickness and L/h ratio.
  Overlap length $d = 5h$

Shear tress $\overline{S}_{13}, H = h$

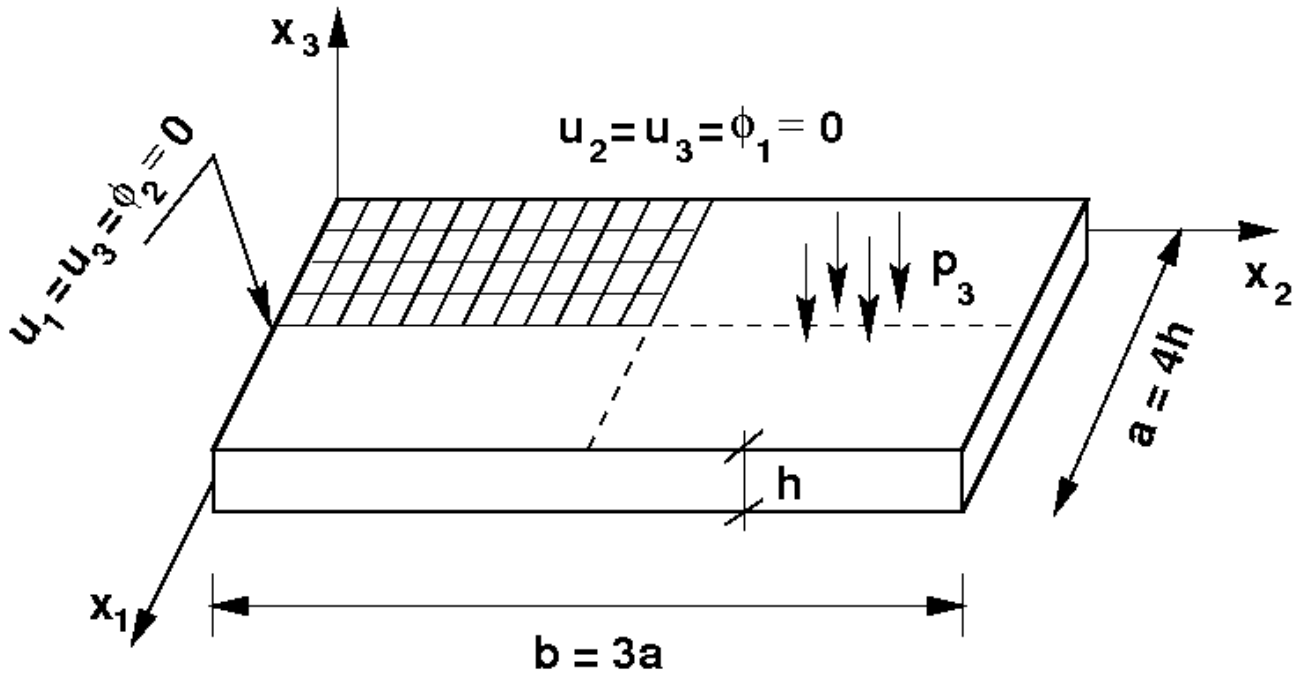


Shear stress $\overline{S}_{13}, H = h/2$



EXAMPLE 3: Initial failure maps due to bending

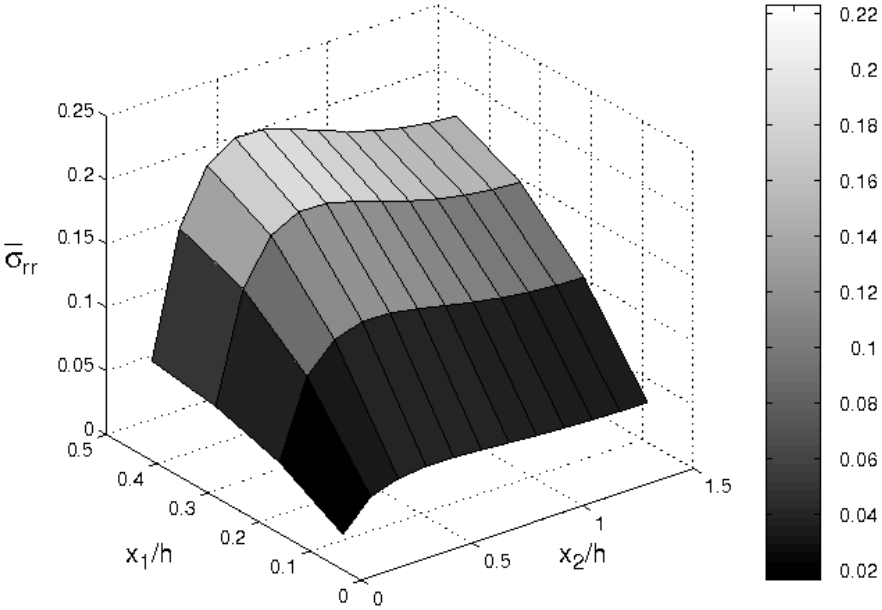
- Evaluation of maximum stresses at fiber matrix interface
- Simple multi-scale analysis combining FEM and the Mori-Tanaka method (Linearly varying play stresses replaced by through-thickness averages)
- Geometry, loading and support conditions



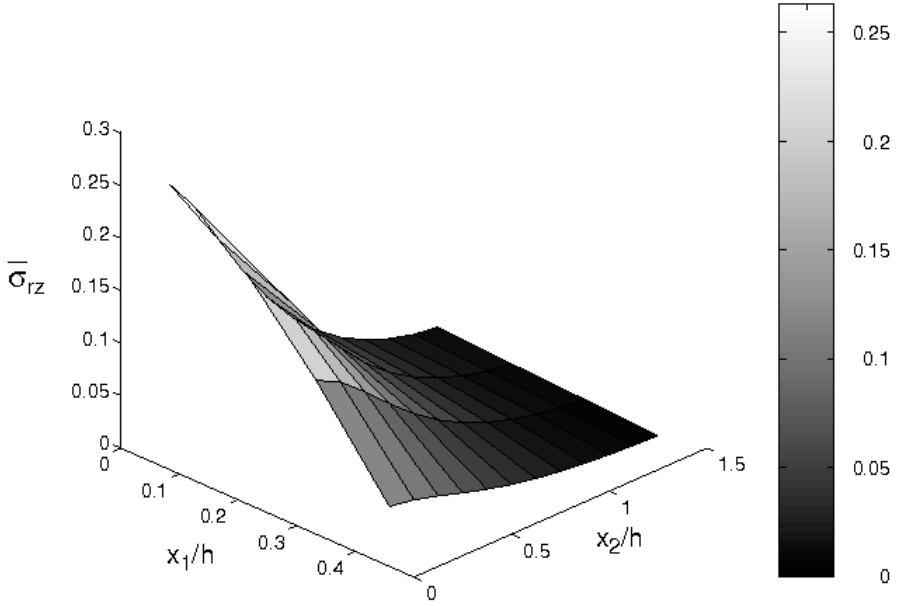
EXAMPLE 3: Initial failure maps due to bending

- Example: distribution of maximum interface stress over the mid-plane of $[0]_8$ SCS6 / Timetal laminated plate
  $(\overline{\sigma}_{zz}, \overline{\sigma}_{rr}, \overline{\sigma}_{\theta\theta}, \overline{\sigma}_{rz}) = \frac{1}{S^2 p_3}(\sigma_{zz}, \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{rz}), \quad S = \frac{a}{h}$

Interface tensile stress $\overline{\sigma}_{rr}$



Interface shear stress $\overline{\sigma}_{rz}$



Eigenstrains as control variables

- Objective: derive optimal distribution of through thickness fiber pre-stress to minimize in-plane stresses of the laminate
- Objective functional

$$\phi = \frac{1}{2}\sum_{i=1}^{\mathrm{N}} \int_{V_i} \left(\boldsymbol{s}_m^i\right)^{\mathrm{T}} \boldsymbol{s}_m^i \,\mathrm{d}V = \frac{1}{2}\sum_{i=1}^{\mathrm{N}} \int_{S_m^i} \varphi_i \left[\boldsymbol{E}_m^0(\boldsymbol{\mu}),\, \overline{\boldsymbol{\kappa}}(\boldsymbol{\mu}),\, \boldsymbol{\mu}\right] \mathrm{d}S$$

- Stationary condition

$$\begin{aligned}\delta\phi \quad = \quad & \sum_{i=1}^{\mathrm{N}} \int_{S_m^i} \sum_{\alpha=1}^{\mathrm{M}} \delta\left(\overline{\boldsymbol{\mu}}_m^i\right)^{\mathrm{T}} \sum_{j=1}^{\mathrm{N}} \frac{\partial\left(\overline{\boldsymbol{\mu}}_m^i\right)^{\mathrm{T}}}{\partial \overline{\boldsymbol{\mu}}_m^{\alpha}} \\ & \left\langle \frac{\partial\left(\overline{\boldsymbol{\mu}}_m^i\right)^{\mathrm{T}}}{\partial \overline{\boldsymbol{\mu}}_m^j} \frac{\partial \varphi_i^{\mathrm{T}}}{\partial \overline{\boldsymbol{\mu}}_m^i} + \frac{\partial\left(\boldsymbol{E}_m^0\right)_i^{\mathrm{T}}}{\partial \overline{\boldsymbol{\mu}}_m^j} \frac{\partial \varphi_i^{\mathrm{T}}}{\partial\left(\boldsymbol{E}_m^0\right)_i} + \frac{\partial \overline{\boldsymbol{\kappa}}_i^{\mathrm{T}}}{\partial \overline{\boldsymbol{\mu}}_m^j} \frac{\partial \varphi_i^{\mathrm{T}}}{\partial \overline{\boldsymbol{\kappa}}_i} \right\rangle \mathrm{d}S\end{aligned}$$

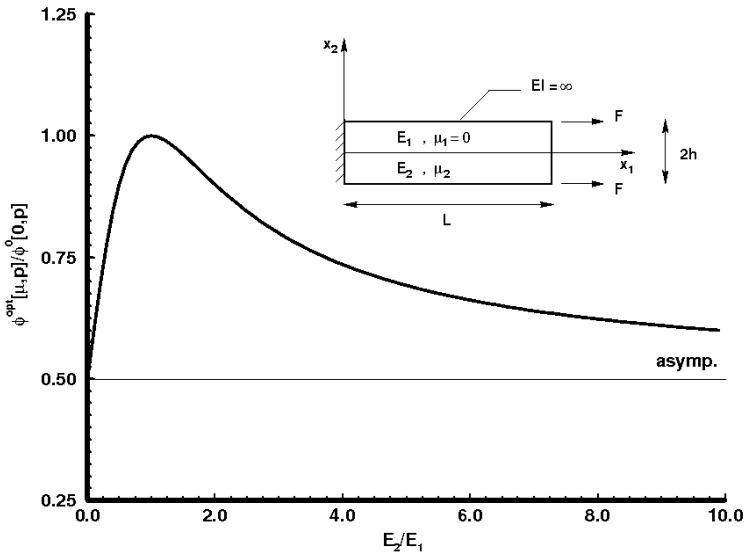
EXAMPLE 1: Optimization of two-layer beam structure

- Objective function

$$\phi = \phi(\boldsymbol{\epsilon}(\boldsymbol{\mu}), \boldsymbol{\mu}, \boldsymbol{F}) = \int_V \left(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} \right) \mathrm{d}V$$

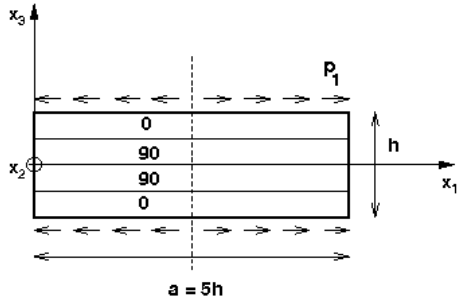
- Stationary solution

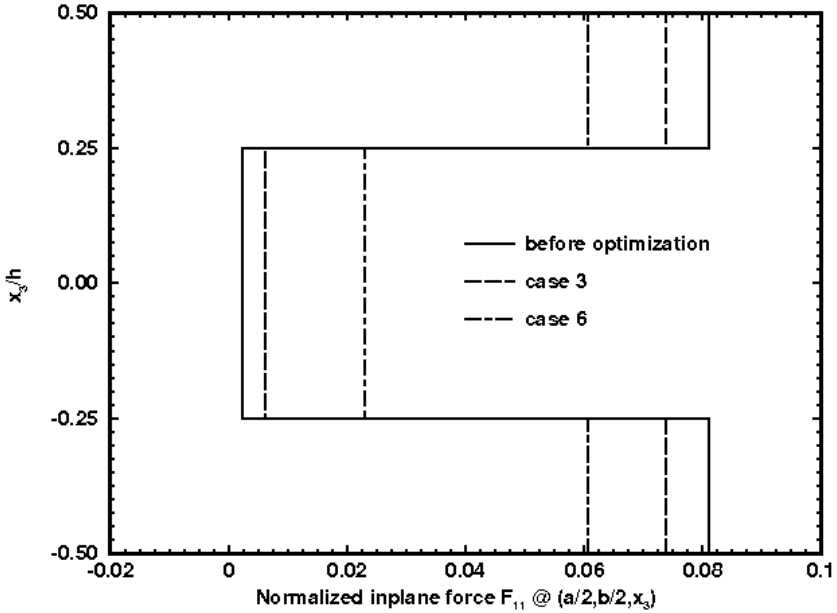
$$\frac{d\phi}{d\mu_2} = \frac{\partial \phi}{\partial \mu_2} + \frac{\partial \phi}{\partial \epsilon} \frac{\partial \epsilon}{\partial \mu_2} = 0 \quad \Longrightarrow \quad \mu_2 = \frac{F}{E_2 h} \left(1 - \frac{E_2}{E_1} \right)$$



EXAMPLE 2: three-ply $[0]_3$ laminate loaded in shear

case/ply	0^0	90^0	90^0	0^0	ϕ^{opt}	ϕ^{opt}/ϕ^0
1	μ_{11}	0	0	μ_{11}	0.567	0.889
2	μ_{22}	0	0	μ_{22}	0.626	0.945
3	μ_{11}, μ_{22}	0	0	μ_{11}, μ_{22}	0.559	0.846
4	0	μ_{11}	μ_{11}	0	0.451	0.682
5	0	μ_{22}	μ_{22}	0	0.662	1.001
6	0	μ_{11}, μ_{22}	μ_{11}, μ_{22}	0	**0.441**	**0.667**





Potential applications and current research effort

- Modeling of delamination of laminated plates using FETI method
- Modeling of laminated glass beams and plates
  - Typically a three-layer laminate consisting of two glass panes bonded by polymeric interface
  - Simplified models based on effective thickness and through-thickness homogenized properties typically do not work $\rightarrow$ calls for layered plate theory
  - Analysis highly dependent on the formulation of constitutive model of interface layer - Maxvell chain model
    - Both small-scale (at the level of material constituents) as well full scale (at the level of laminate) experiments are needed to tune the properties of the Maxwell model